(Neo)classical transport

Classical transport in fluid picture (cylinder):

MHD-eq: $\nabla p = \vec{j} \times \vec{B}$

Ohm's law: $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$

$$v_\perp = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{\eta}{B^2}\vec{j}\times\vec{B} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{\eta\nabla p}{B^2}$$

Collisions (resistivity) lead to radial velocity!

Consider diffusive particle flux (T=const):

$$\Gamma = n\vec{v} = -D\nabla n$$

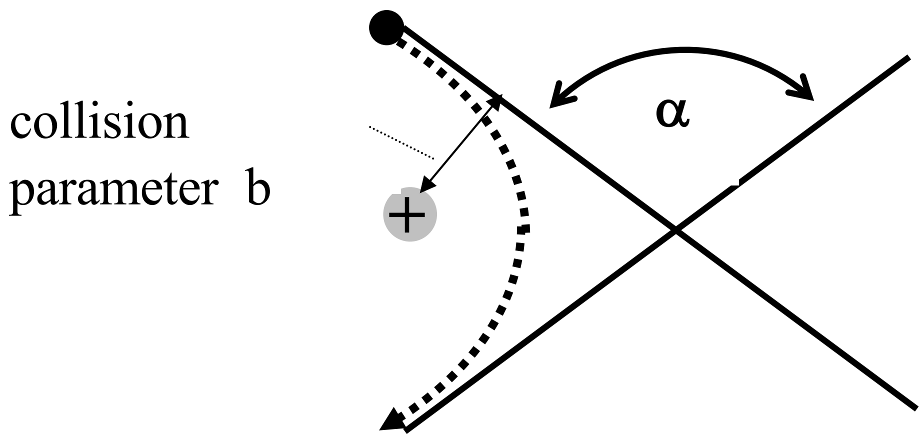
$$\vec{v}_\perp = -\eta\frac{2k_B T\nabla n}{B^2} = -\frac{\nu m_e 2k_B T}{ne^2 B^2}\nabla n = -\frac{1}{n}r_L^2\nu\nabla n \longrightarrow D = r_L^2\nu$$

$$\eta_{\parallel} = \frac{m_e}{n_e \cdot e^2}\cdot \nu_{stoß}$$

$$r_{ge} = \frac{m_e v_\perp}{e\cdot B}$$

$$v_{\mathrm{th}} = \sqrt{\frac{k_B T}{m}}.$$

Coulomb collisions:momentum exchange



example:
e-collision with ion (charge Z)

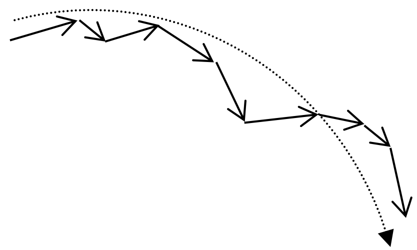
cross section for scattering by 90º:

$$\sigma_{90} = \pi . b_{90}^2 = \frac{\pi \cdot Z^2 \cdot e^4}{(4\pi\varepsilon_0)^2 \cdot 4 \cdot (W_{kin})^2}$$

$\Rightarrow$ Coulomb cross section depends strongly on particle energy : ~ $1/W_{kin}^2$

Coulomb collisions:momentum exchange

scattering by 90º via many small-angle collisions



correction of cross section: $$\frac{\sigma_{90-effektiv}}{\sigma_{90-direkt}} = 8 \cdot \ln \frac{\lambda_D}{b_{90}} = 8 \cdot \ln \Lambda \quad \geq 100$$

Coulomb collisions:momentum exchange

$$\sigma_{ei} = 8\ln\lambda\,\sigma_{90} = 8\ln\lambda\,\frac{\pi\cdot Z^2\cdot e^4}{(4\pi\varepsilon_0)^2\cdot 4\cdot (W_{kin})^2}$$

with: W_{kin}=3/2kT_e and
v = (3·kT_e/m_e)$^{1/2}$

$$\frac{m}{2}\mathrm{v}_{eff}^2 \stackrel{def}{=} \frac{3}{2}kT$$

$$\sigma_{ei} = 8\ln\lambda\,\sigma_{90} = 8\ln\lambda\,\frac{\pi\cdot Z^2\cdot e^4}{(4\pi\varepsilon_0)^2\cdot m_e^2\cdot v_e^4} = 8\ln\lambda\,\frac{\pi\cdot Z^2\cdot e^4}{(4\pi\varepsilon_0)^2\cdot (3k_BT)^2}$$

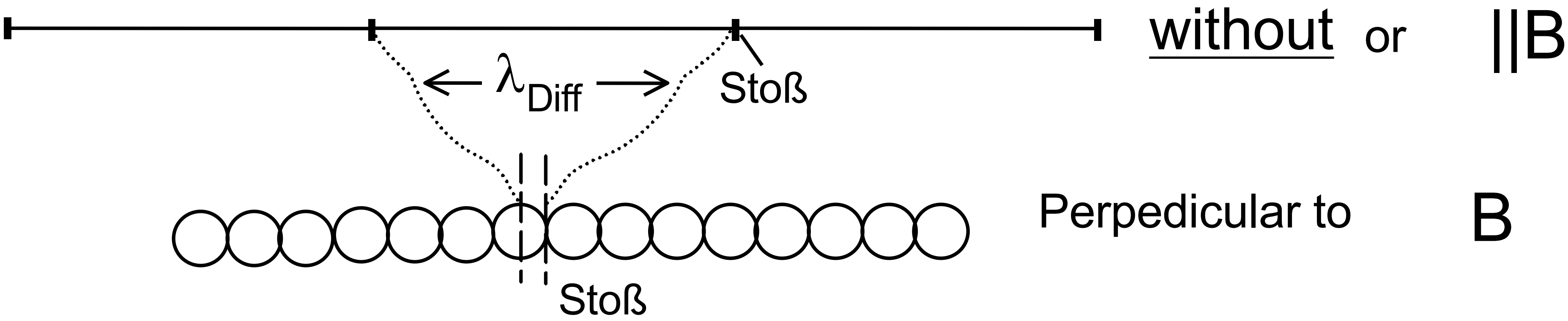
collision time for electron-ion-collisions:

$$\tau_{\mathrm{ei}} = 1/\nu_{\mathrm{ei}} \qquad \nu_{ei} = n_i\cdot\langle\sigma_{ei}\,v_e\rangle \sim n_i\sigma_{ei}\langle v_e\rangle$$

$$\nu_{ei} = 8\ln\lambda\,\frac{n_i\pi\cdot Z^2\cdot e^4}{(4\pi\varepsilon_0)^2\cdot (3k_BT)^{3/2}\sqrt{m_e}} = \frac{1}{\tau_{ei}}$$

Classical transport coefficients (in particle picture)

- Estimate transport coefficients: Δt from collision frequency ν



Collision frequencies (90°)

$$\nu_{ee} \approx \nu_{ei} \propto \frac{ne^4}{\sqrt{m_e} T_e^{3/2}}$$

$$\nu_{ie} = \left(\frac{m_e}{m_i}\right)\nu_{ee}$$

$$\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_e}{Ti}\right)^{3/2}\nu_{ee}$$

Classical transport coefficients in plasma

- Δt from collision frequency ν

$$\nu_{ee} \approx \nu_{ei} \propto \frac{ne^4}{\sqrt{m_e} T_e^{3/2}}$$

$$\nu_{ie} = \left(\frac{m_e}{m_i}\right) \nu_{ee}$$

$$\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{T_e}{Ti}\right)^{3/2} \nu_{ee}$$

- typical perpendicular transport length is the Larmor radius:

$$\Delta x = r_L = \frac{\sqrt{2mkT}}{eB}$$

- no particle transport between particles of same species:

$$\vec{r}_g = \frac{m}{qB^2} \vec{v} \times \vec{B} = \frac{\vec{p} \times \vec{B}}{qB^2}. \qquad \Delta \vec{r}_1 = \frac{\Delta \vec{p}_1 \times \vec{B}}{eB^2} = -\frac{\Delta \vec{p}_2 \times \vec{B}}{eB^2} = -\Delta \vec{r}_2$$

(only exchange of particle position for same charge, transport for e-i collisions)

Classical transport coefficients in plasma

•Δt from collision frequency ν

$$\nu_{ee} \approx \nu_{ei} \propto \frac{ne^4}{\sqrt{m_e} T_e^{3/2}}$$

$$\nu_{ie} = \left(\frac{m_e}{m_i}\right) \nu_{ee}$$

$$\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{T_e}{Ti}\right)^{3/2} \nu_{ee}$$

• typical perpendicular transport length is the Larmor radius:

$$\Delta x = r_L = \frac{\sqrt{2mkT}}{eB}$$

• for $e - i$ collisions: transport is *ambipolar:*

$$D_{e,class} = \nu_{ei} r_{L,e}^2 = \nu_{ie} \frac{m_i}{m_e} r_{L,e}^2 = \nu_{ie} r_{L,i}^2 = D_{i,class}$$

Classical transport coefficients in plasma

Classical heat conductivity:also collisions between particles of the same Species contribute to heat transport:

$$\kappa_{ii} = n_i r_{Li}^2 \nu_{ii} \qquad \chi = \kappa / n$$

$$\chi_{ee} \approx \chi_{ei} \approx \chi_{ie} \approx \sqrt{\frac{m_e}{m_i}} \chi_{ii}$$

- classical heat conduction: $\chi_i \approx 40 \chi_e$

•Typical number for ions: $10^{-4} m^2 / s$

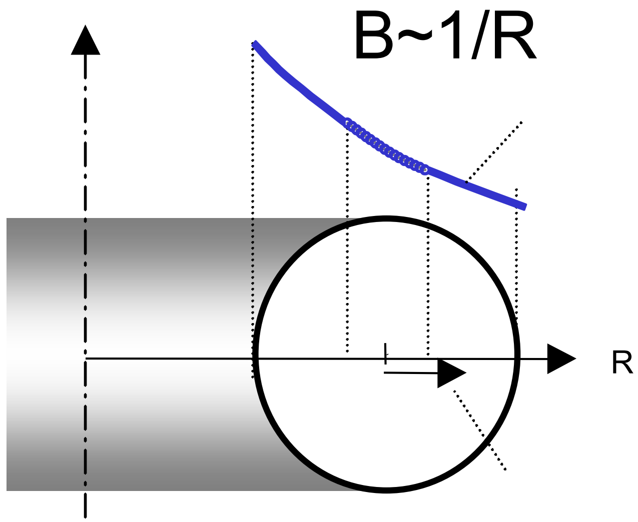
- experimentally found: $1\, m^2 / s$, and $\chi_e \approx \chi_i$

Neoclassical transport: (torus)

- change due to toroidal effects $\varepsilon = r / R$

- Along magnetic field lines B is not constant: mirror

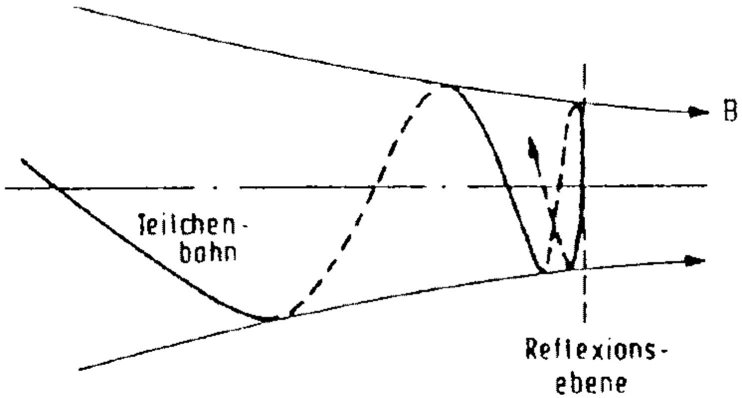
- Depending on $v_{\parallel} / v_{\perp}$, particles can be trapped

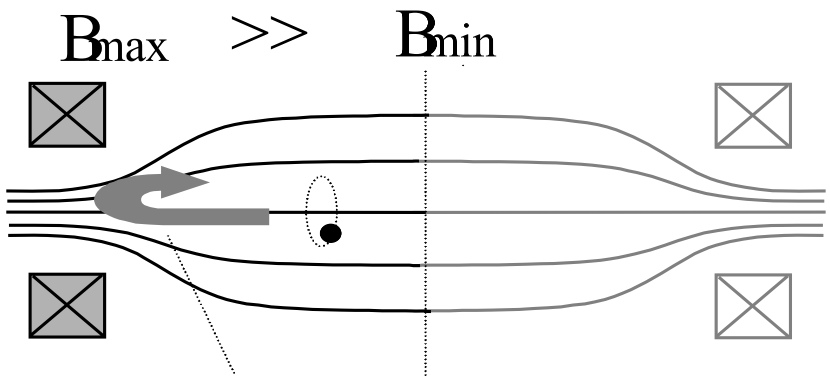


Neoclassical transport (transport in a torus)

Magnetic moment is invariant: $\mu = \frac{m v_\perp^2}{2B}$

If total energy is constant, parallel energy has to decrease if B increases up to $v_{||}=0$ (reflection)



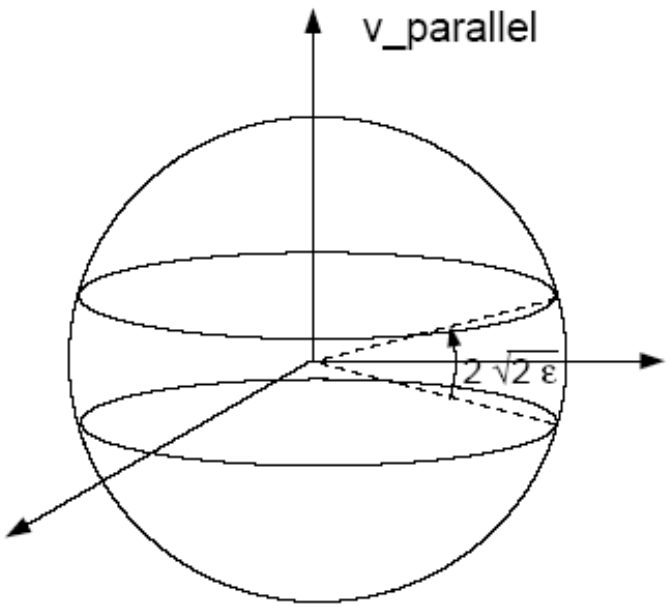


Mirror condition: $$\frac{v_{\parallel}^2(B_{\min})}{v_{\perp}^2(B_{\min})} < \frac{B_{\max}}{B_{\min}} - 1$$

Mirror condition for magnetic surface r:

$$\frac{B_{max}}{B_{min}} - 1 = \frac{B_0(R_0 + r)}{B_0(R_0 - r)} - 1 = \frac{1 + r/R_0}{1 - r/R_0} - 1 = \frac{2r/R_0}{1 - r/R_0}$$

ε/R<<1: $$\left|\frac{v_{\parallel}}{v_{\perp}}\right| < \sqrt{2\epsilon}$$

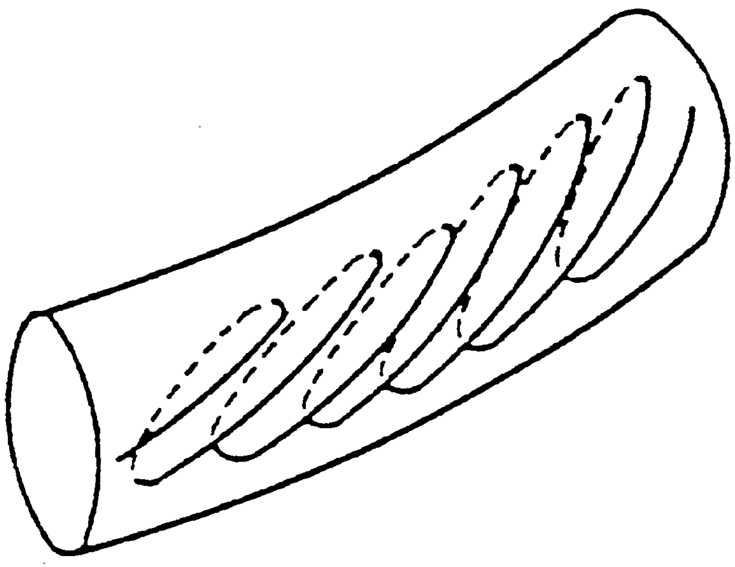


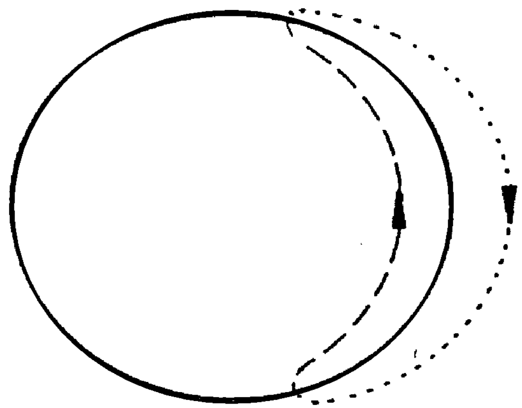
Neoclassical Transport (transport in torus)

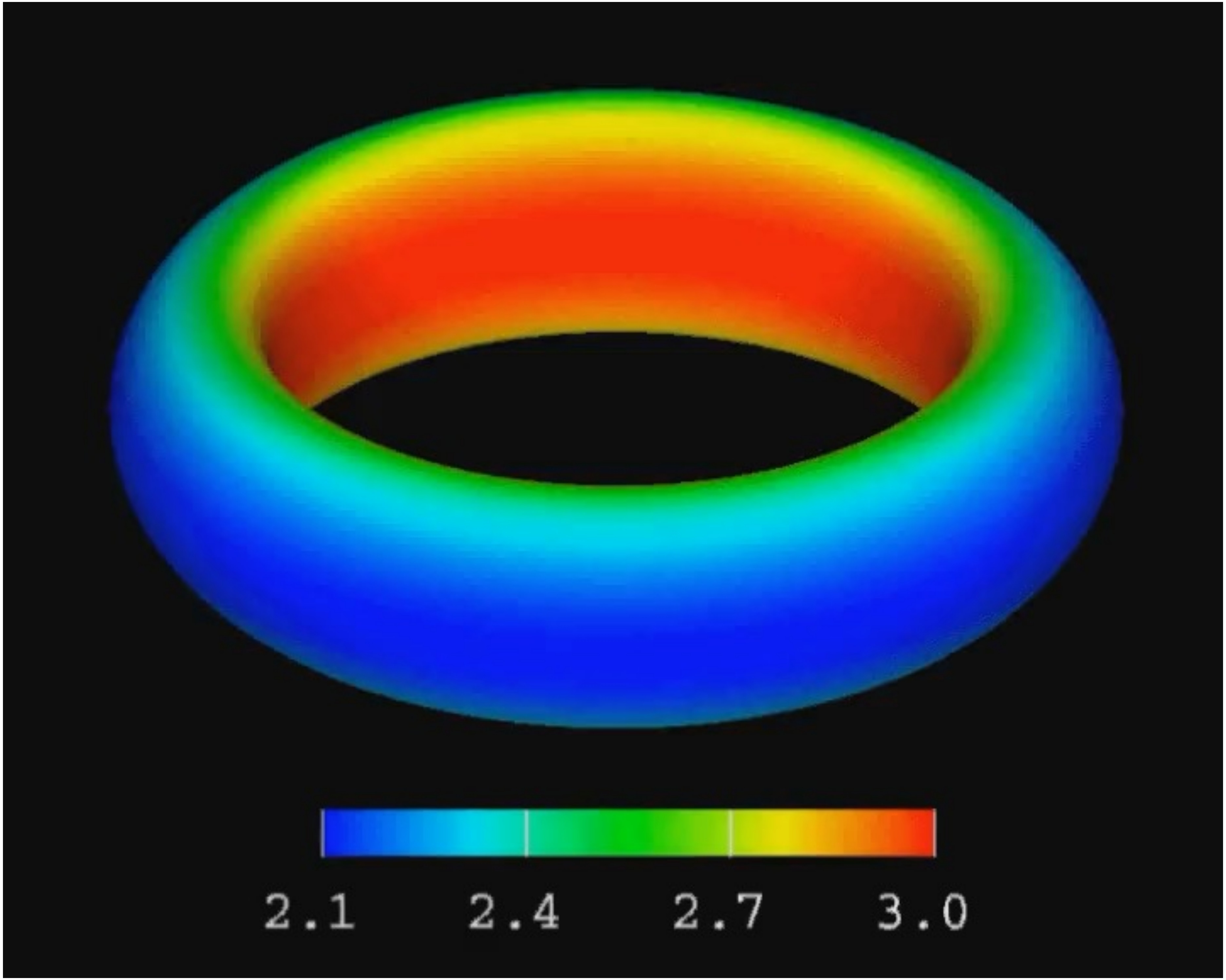
• depending on $v_{||} / v_{\perp}$ particles are trapped

• drift in inhomogeneous magnetic field $$\vec{v}_D = \frac{m}{qB^3}\left(v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2\right)\vec{B} \times \nabla B$$

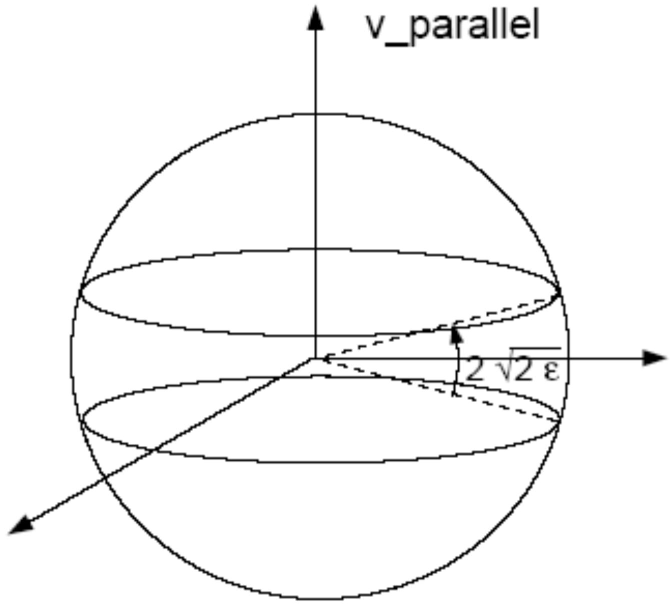
• orbit of trapped particles - banana orbit






2.1
2.4
2.7
3.0

Fraction of trapped particles



Integration over velocity space (at constant velocity)

$$\tan\theta = \frac{v_\parallel}{v_\perp} \qquad \left|\frac{v_\parallel}{v_\perp}\right| < \sqrt{2\epsilon}$$

$$\varepsilon \ll 1: \tan\sqrt{2\varepsilon} \simeq \sqrt{2\varepsilon}$$

$$\frac{n_t}{n} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-\sqrt{2\epsilon}}^{\sqrt{2\epsilon}} \cos\theta d\theta = \frac{1}{2}\Big(\sin\sqrt{2\epsilon} - \sin\left(-\sqrt{2\epsilon}\right)\Big) \approx \sqrt{2\epsilon}$$

$$\varepsilon \ll 1: \sin\sqrt{2\varepsilon} \simeq \sqrt{2\varepsilon}$$

Fraction of trapped particles

$$\frac{n_t}{n} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-\sqrt{2\epsilon}}^{\sqrt{2\epsilon}} \cos\theta\, d\theta = \frac{1}{2}\Big(\sin\sqrt{2\epsilon} - \sin(-\sqrt{2\epsilon})\Big) \approx \sqrt{2\epsilon}$$

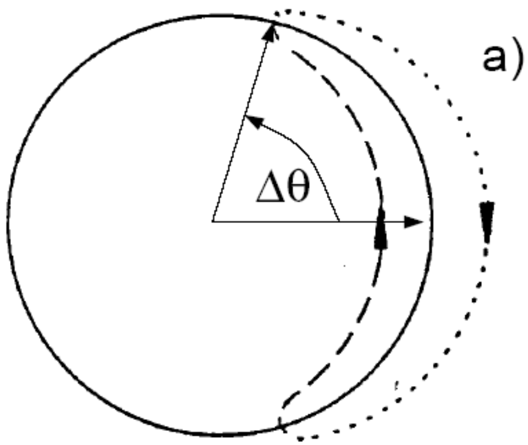
Estimate banana width:

i.e. deviation from magnetic surface (assume $v_{||}$ small):

$$\vec{v}_D = \frac{m}{qB^3}\left(v_{\|}^2 + \frac{1}{2}v_{\perp}^2\right)\vec{B}\times\nabla B = \frac{m}{eBR}\left(v_{\|}^2 + \frac{1}{2}v_{\perp}^2\right) \approx \frac{m}{2eBR}v_{\perp}^2$$

Banana width~ $v_D \Delta t$ (Δt :time to sample a banana orbit)

Time to complete a banana orbit: $v_{||}$ x L (length of a field line)



$$L \approx R\Delta\phi = qR\Delta\theta$$

$$\Delta t = L/v_{||} = \frac{qR\Delta\theta}{v_{||}}$$

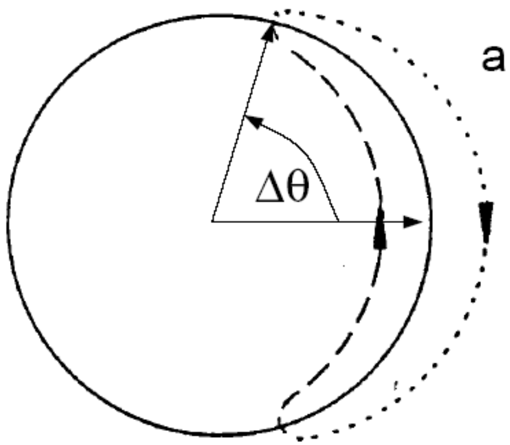
Banana width: $$w_B = v_D \Delta t = \frac{mv_\perp}{eB}\frac{q}{2}\frac{v_\perp}{v_{||}}\Delta\theta = r_L \frac{q}{2}\frac{v_\perp}{v_{||}}\Delta\theta$$

grad B drift: $$v_D = \frac{mv_\perp^2}{2eBR}$$

$$r_{ge} = \frac{m_e V_\perp}{e \cdot B}$$

Banana width~ $v_D\Delta t$ (Δt :time to sample a banana orbit)

Time to complete a banana orbit: $v_{||}$ x L (length of a field line)



$$L \approx R\Delta\phi = qR\Delta\theta$$

$$\Delta t = L/v_{||} = \frac{qR\Delta\theta}{v_{||}}$$

Banana width: $$w_B = v_D\Delta t = \frac{mv_\perp}{eB}\frac{q}{2}\frac{v_\perp}{v_{||}}\Delta\theta = r_L\frac{q}{2}\frac{v_\perp}{v_{||}}\Delta\theta$$

Maximal banana width: $\Delta\vartheta=\pi$, and $v_{||}/v_\perp = \sqrt{2\epsilon}$

$$w_B = r_L\frac{\pi}{2\sqrt{2}}\frac{q}{\sqrt{\epsilon}} \approx r_L\frac{q}{\sqrt{\epsilon}}$$

Neoclassical Transport (Transport in a torus)

- Banana width: $r_B = \frac{r_L q}{\sqrt{\varepsilon}}$
- number of trapped particles: $n_t = \sqrt{2\varepsilon}$
- effective collision frequency (trapped $\leftrightarrow$ passing): $\nu_{eff} = \nu_c / (2\varepsilon)$
- D_{neo} by random walk with $1/\nu_{eff}$ and r_B for n_t particles:

$$D_{neo} = r_B^2 \nu_{eff} \frac{n_t}{n}$$

$$D_{neo} = \sqrt{2\varepsilon} r_B^2 \nu_{eff} = \frac{q^2}{\varepsilon^{3/2}} D_{class}$$

- May increase D, χ up to two orders of magnitude:

  χ_i 'only' wrong by factor 3-5

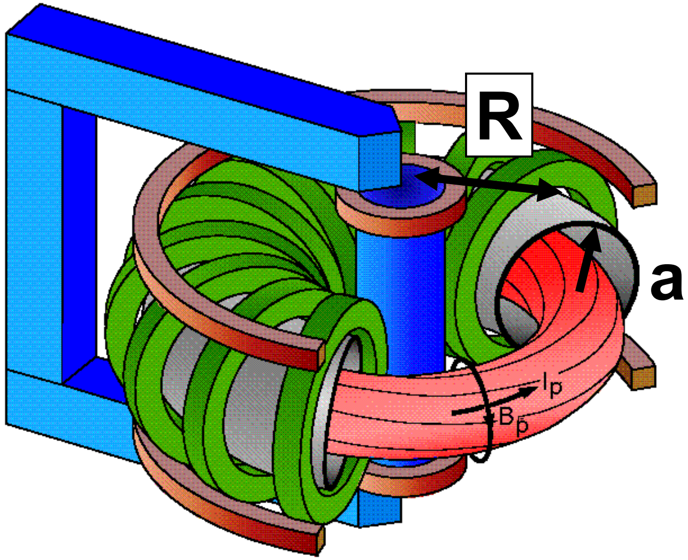
  D, χ_e still wrong by up to two orders of magnitude!

Fusion power plant needs to be much larger than originally thought (turbulent transport)

<u>Experimental result:</u>

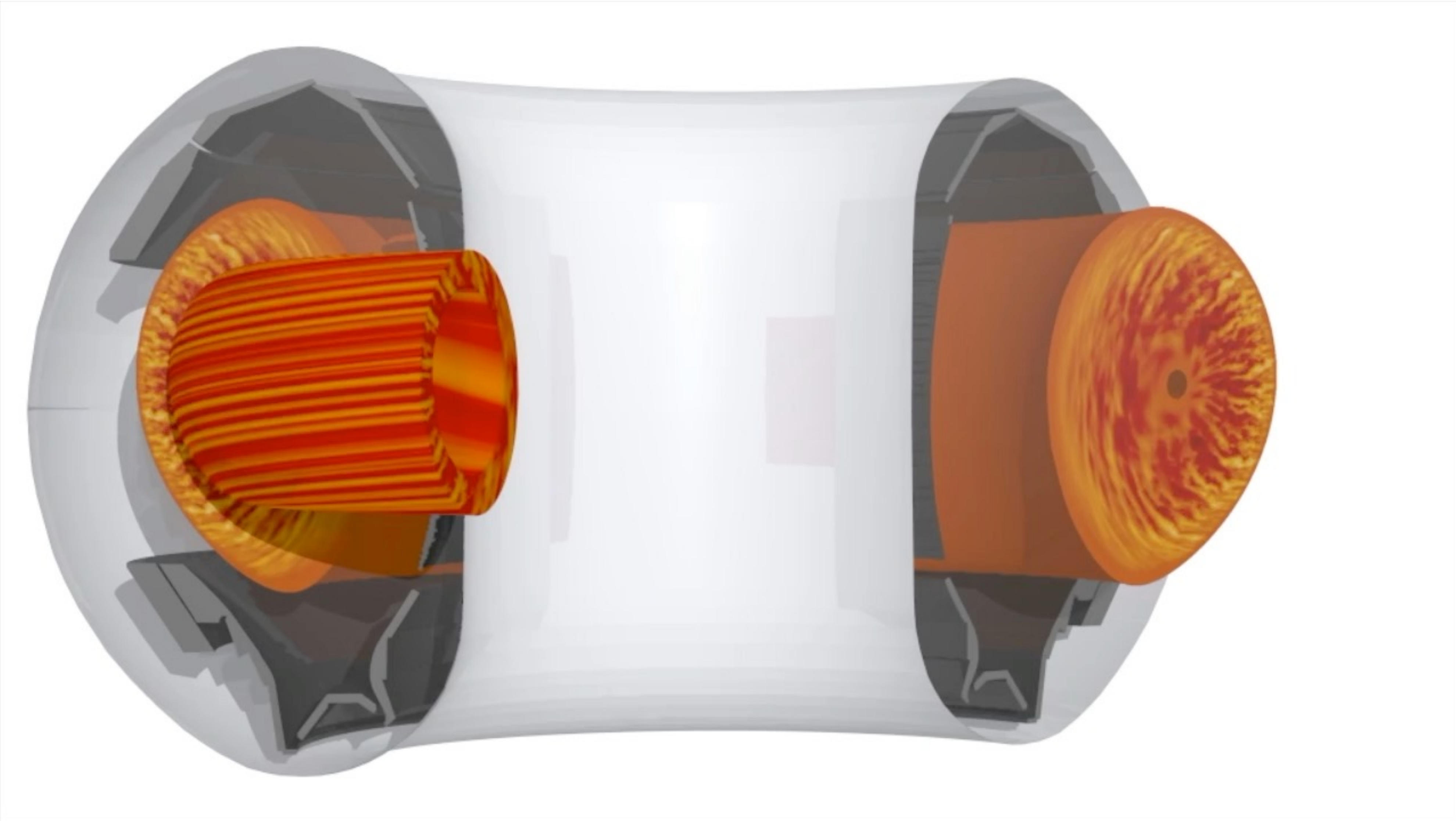
- Anomalous transport:
  χ, D $\approx$ a few m^2/s

Transport increases with
Heating power



- Tokamaks: Ignition expected for R = 8 m

Turbulent transport (next lecture)



Diffusion from „random walk“ consideration:

(characteristic stepsize w_B, collision time: $1/\nu_{eff}$)

$$D_{neo} = w_B^2 \nu_{eff} \frac{n_t}{n} = r_L^2 \frac{q^2 \nu}{2\epsilon^2} \sqrt{2\epsilon} \approx \frac{q^2}{\epsilon^{3/2}} D_{klass}$$

This result is only valid if particles can complete the banana orbits between the collisions often enough; normalise collision time to banana-orbit time:

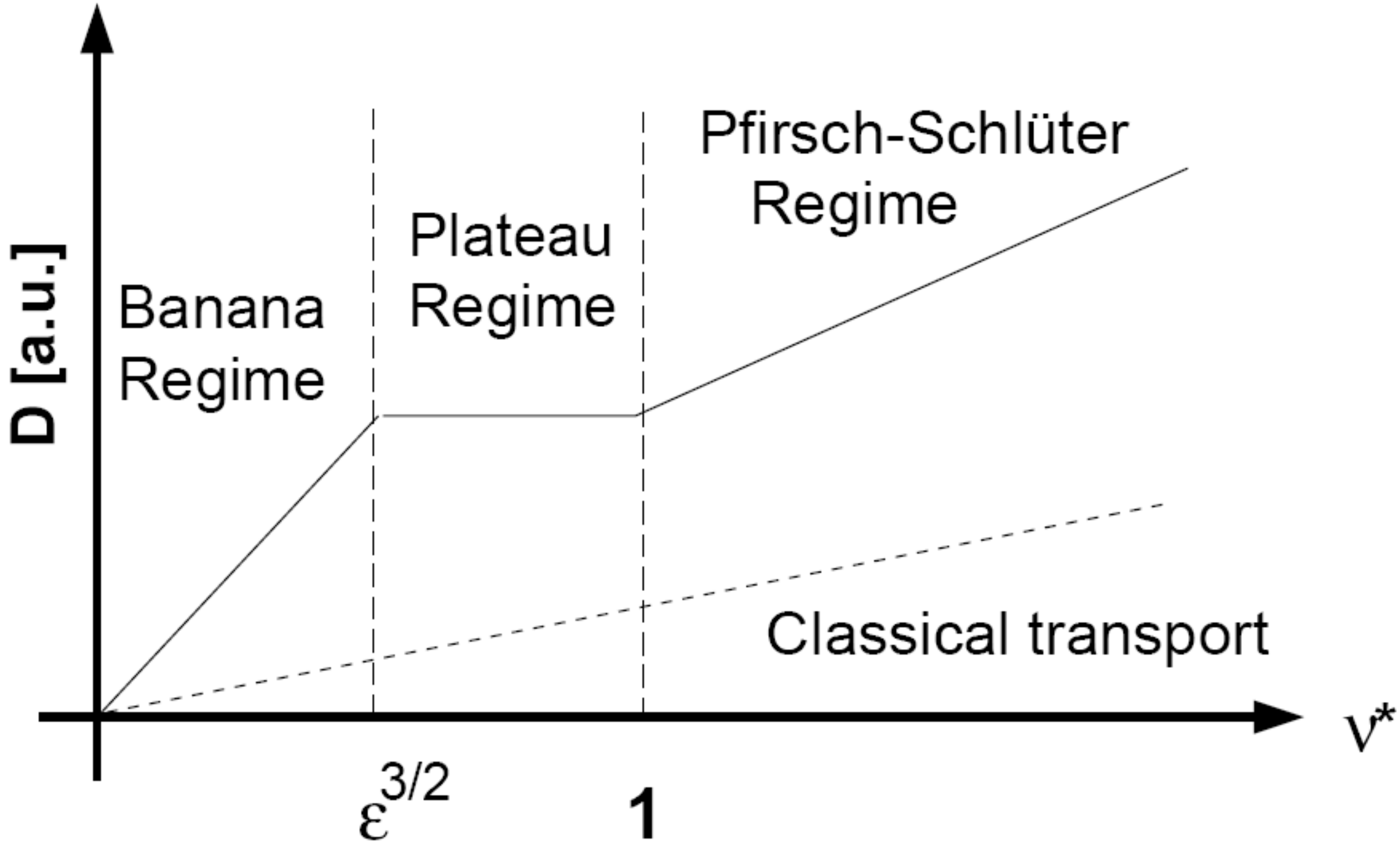
$$\nu^* = \nu_{eff} \Delta t = \frac{\nu q R}{2\epsilon v_{||}}$$

$\nu^* < \varepsilon^{3/2}$: $D_{ban} = D_{klass} q^2 / \epsilon^{3/2}$

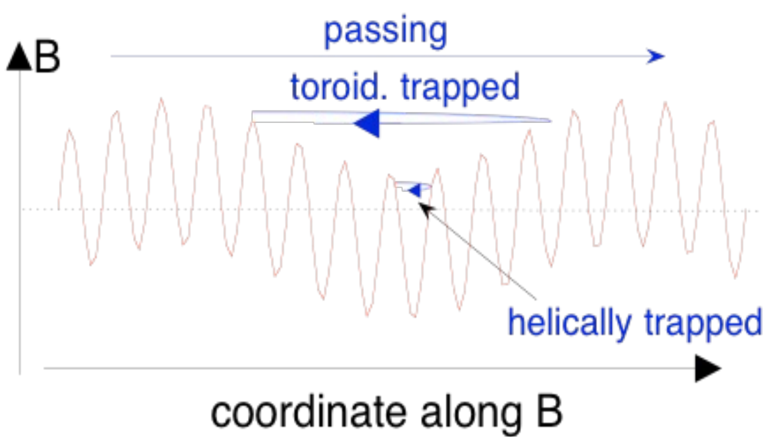
$\nu^* > 1$: $D_{PS} = q^2 r_L^2 \nu = q^2 D_{klass}$

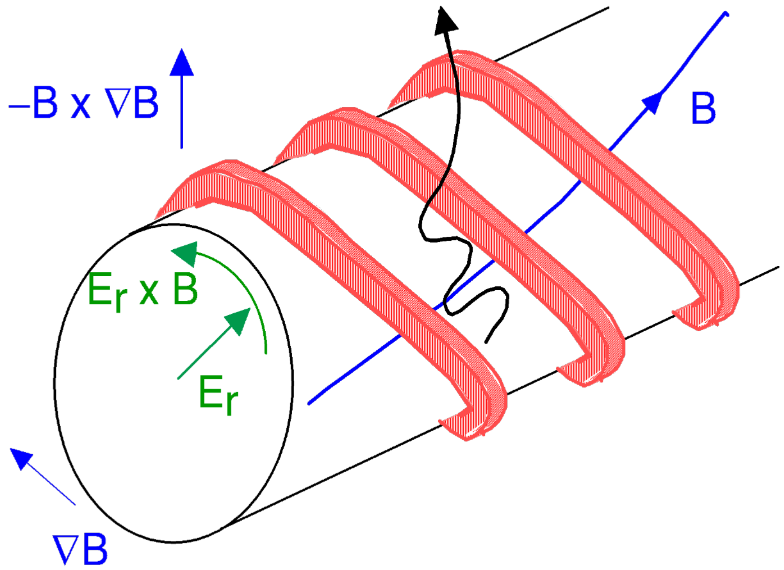
$\varepsilon^{3/2} < \nu^* < 1$: $D_{Plat} = \frac{v_{th} r_L^2 q}{R}$

Neoclassical diffusion coefficients dependence on collisionality



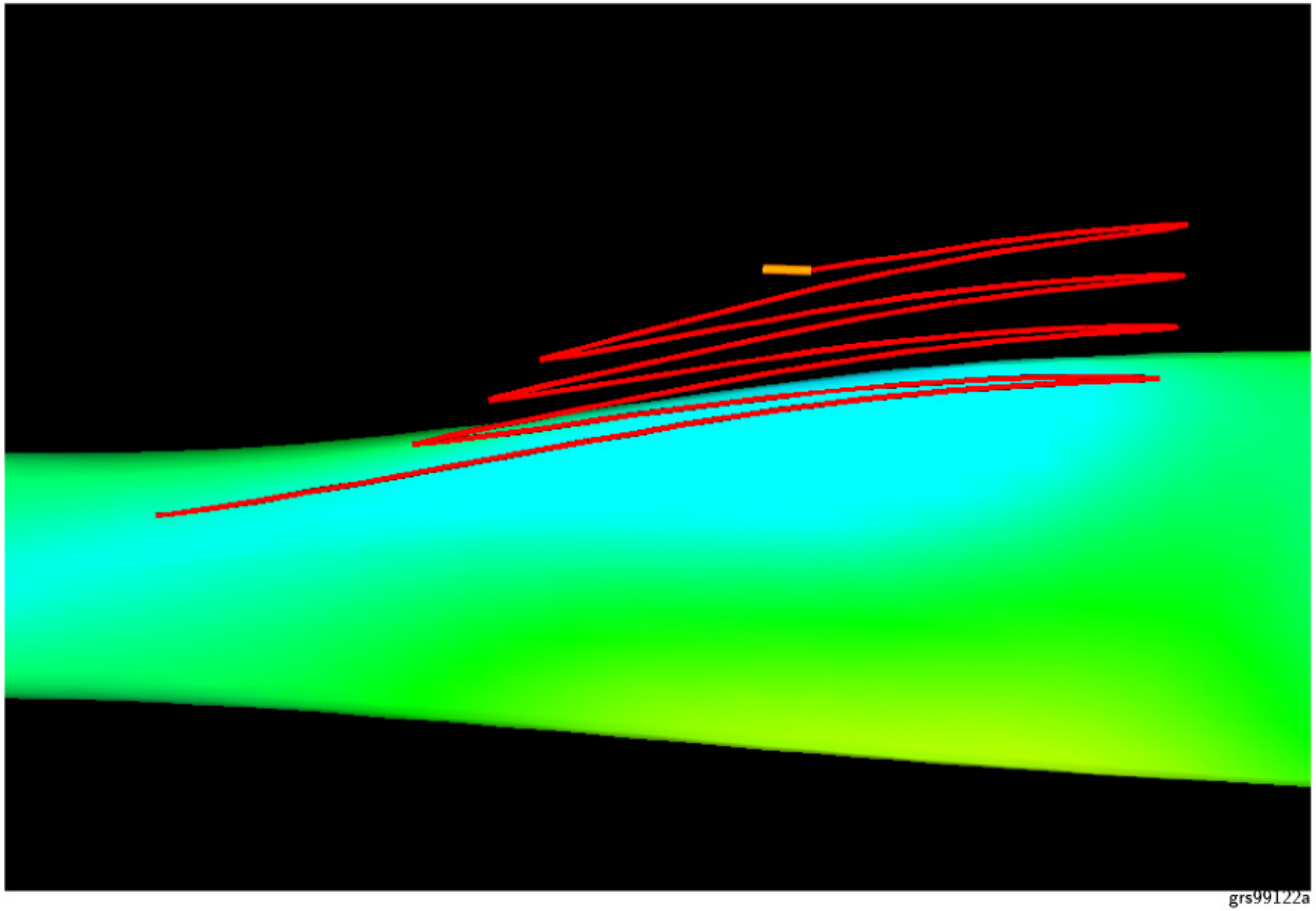
Neoclassical Transport in Stellarators



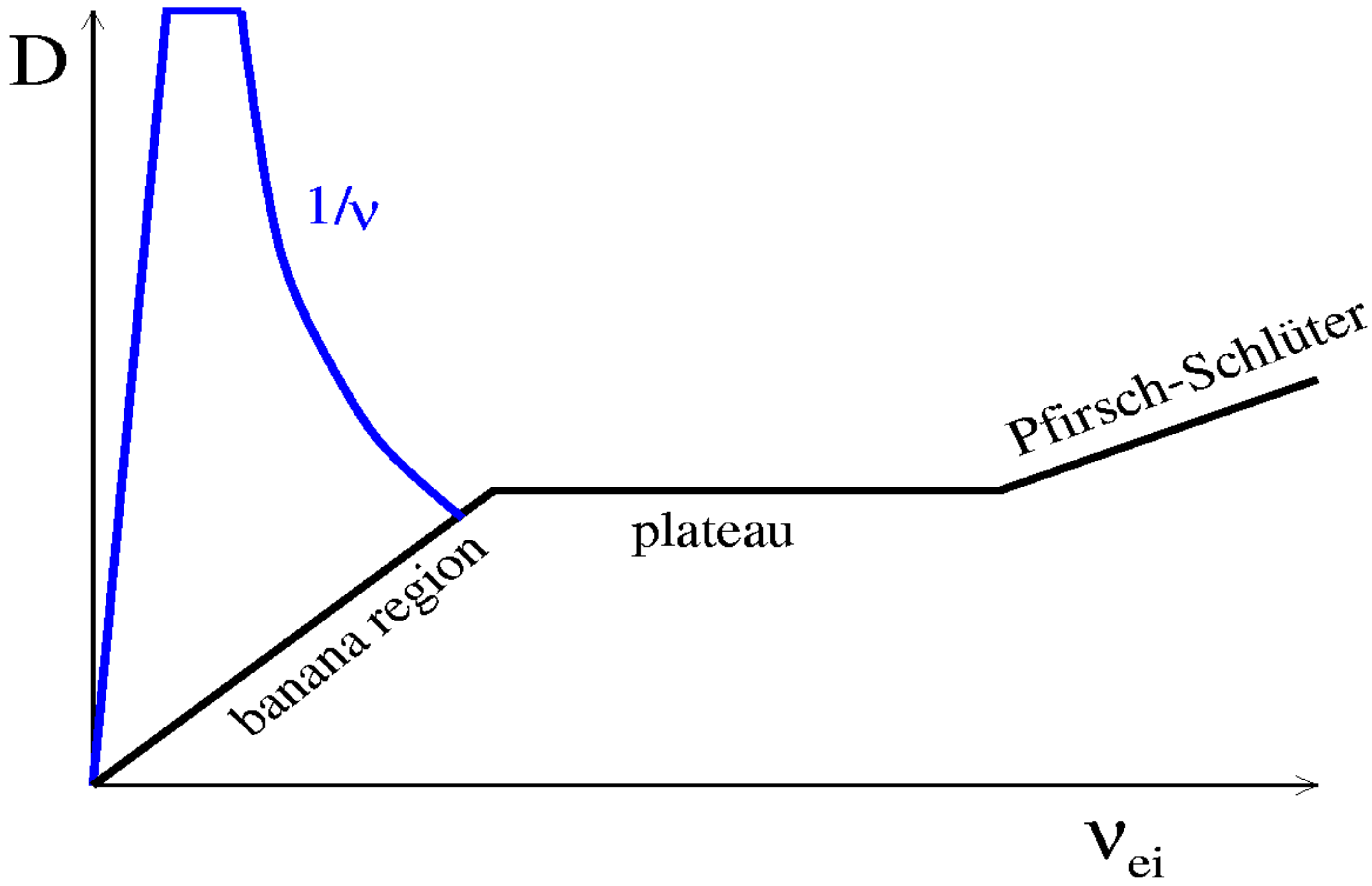


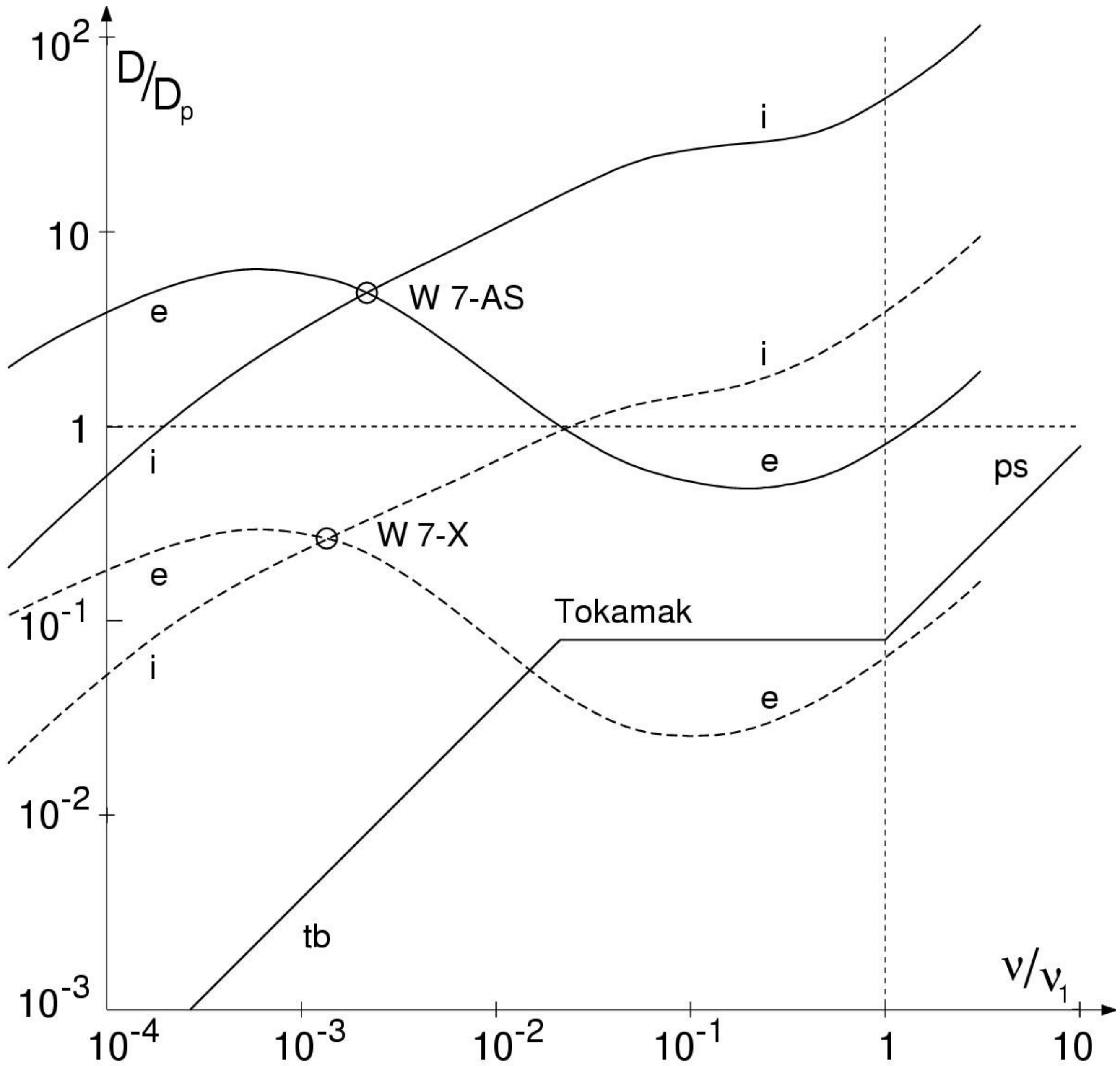
In 3d geometry drift orbits are usually not on a closed surface, in general radial outward motion

Collision-free trapped particles are usually lost in stellarators



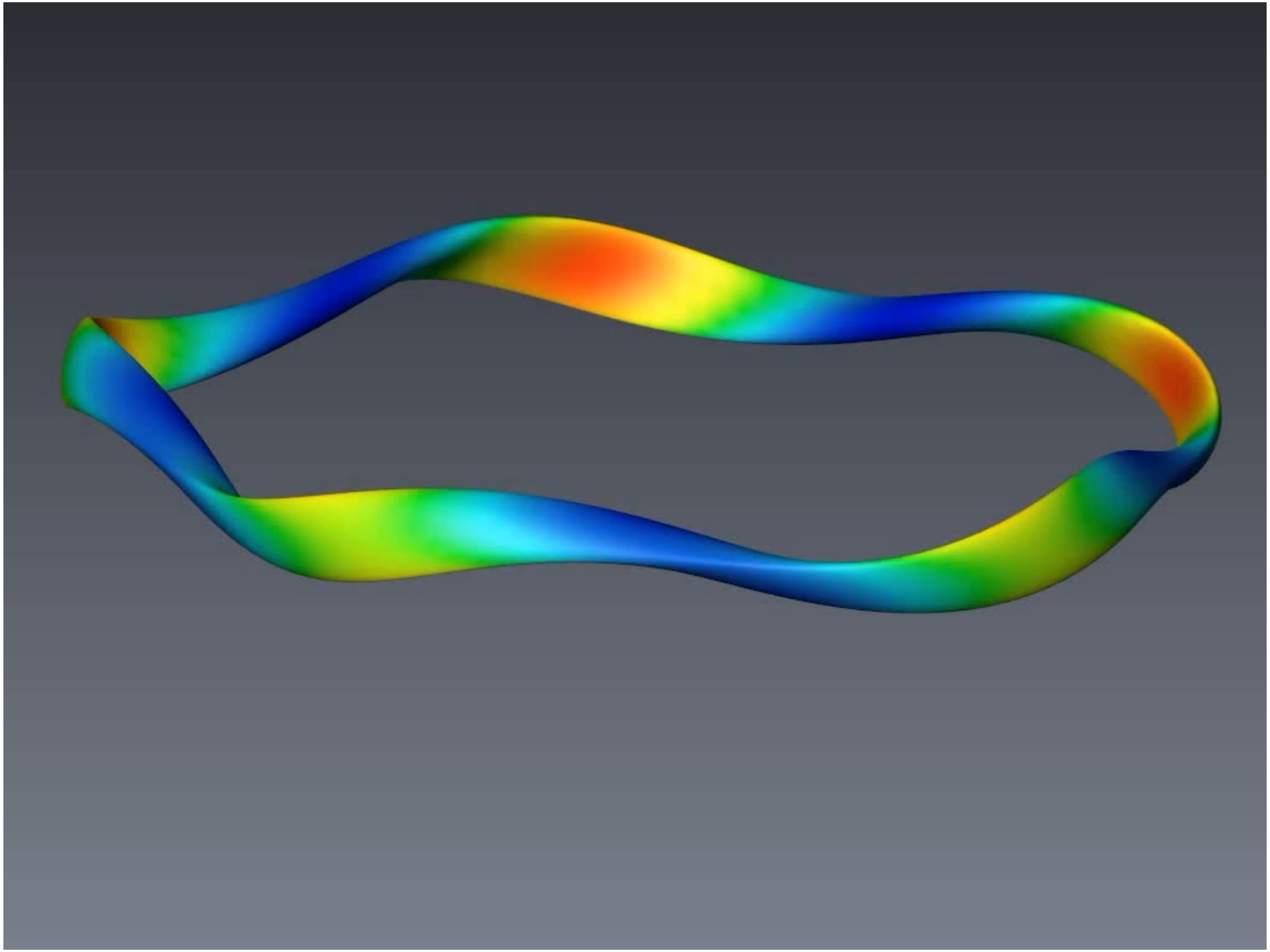
Diffusion coefficient versus collision frequency





10^2
D/D_p
10
1
10^{-1}
10^{-2}
10^{-3}
i
e
W 7-AS
i
e
i
ps
W 7-X
e
Tokamak
i
e
tb
ν/ν_1
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10

W7-X : Optimized neoklassical transport

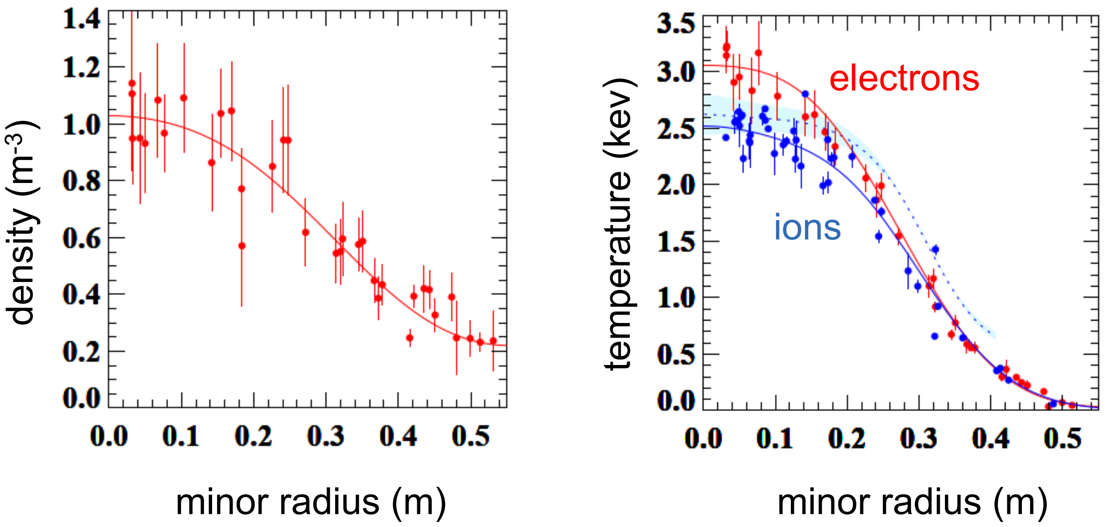


Optimization was successful!

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

Consider record discharge (with partially suppressed turbulent transport, only transiently after injection of pellets)

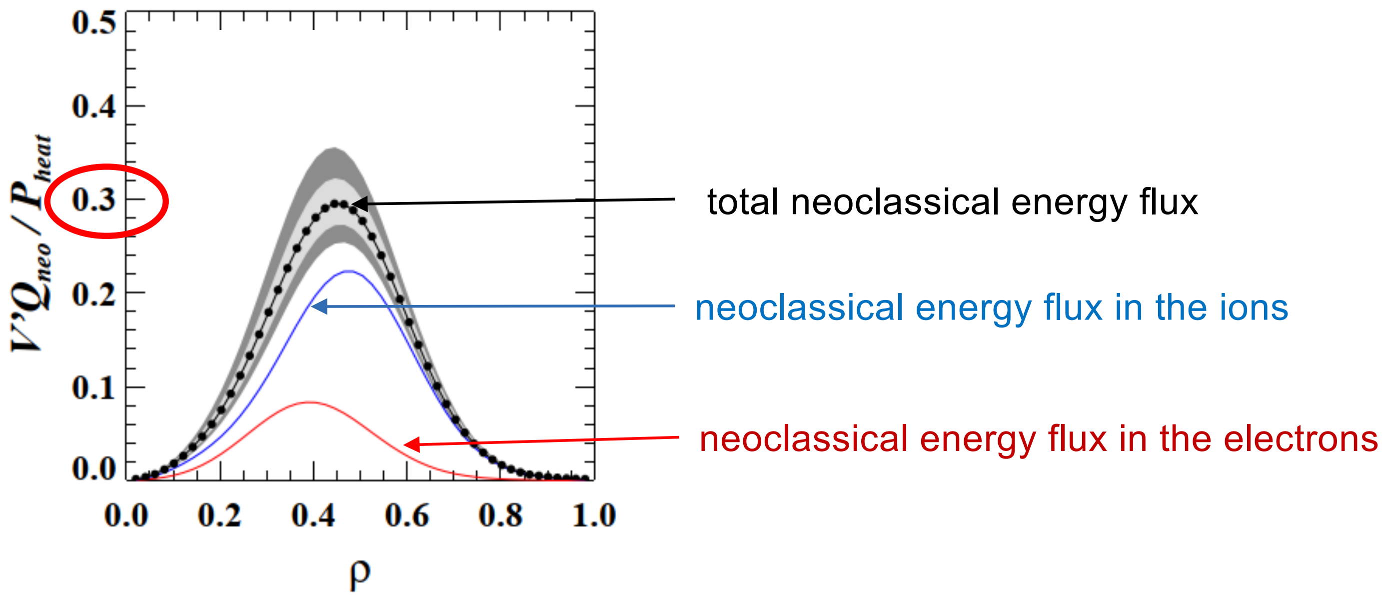
- Peaked density profiles
- Electron and ion temperaturs equal (although electron heating only) (P_{ECRH} = 4.5 MW)



Neoclassical energy flux

Energy flux normalized to heating power as function of plasma minor

(2/3 of the losses are not neoclassical!).



Comparison to other stellarators

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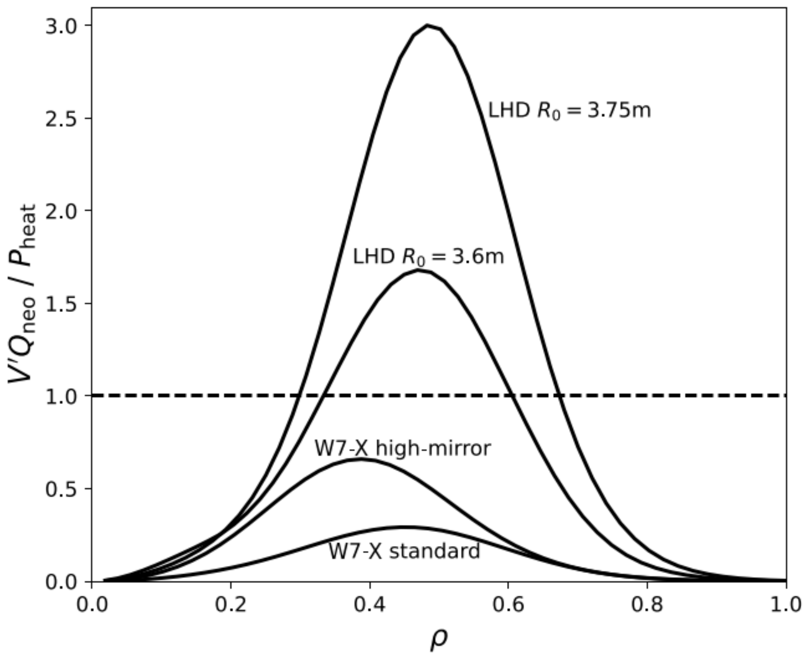
Experiments in W7-X standard configuration

- For comparison: calculated neoclassical energy fluxes for same assumed density and temperatur profiles for different W7-X configurations or stellarators (LHD)
- In non-optimized configurations/stellarators neoclassical energy „losses“ larger than total heating power, i.e. measured high temperatures would not be possible at given heating power



Beidler et al, Nature (2021)

Proof of neoclassical optimization!

Neoclassical effects on plasma current

Correction of conductivity due to trapped particles

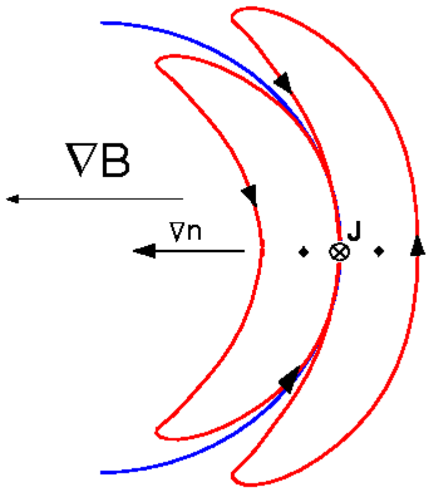
Density of freely moving particles in toroidal direction reduced: $n_e(1-(n_t/n_e))$

Increased collisionality due to momentum transfer between trapped and passing particles: $\nu = \nu_{ei} + \nu_{ee,t} \approx \nu_{ei}(1+n_t/n_e)$

Conductivity reduced compared to Spitzer:

$$\sigma_{Neo} = \sigma_{Sp}\frac{1-n_t/n_e}{1+n_t/n_e} \approx \sigma_{Sp}(1-\frac{n_t}{n_e})^2 = \sigma_{Sp}(1-\sqrt{2\epsilon})^2$$

Banana current



assumption: T=const

Parallel current due to density gradient of trapped particles:

$$j_{gef} \approx e(n_1 - n_2)\sqrt{2\epsilon} v_{||} \sim (n_1 - n_2)\epsilon v_{th}$$

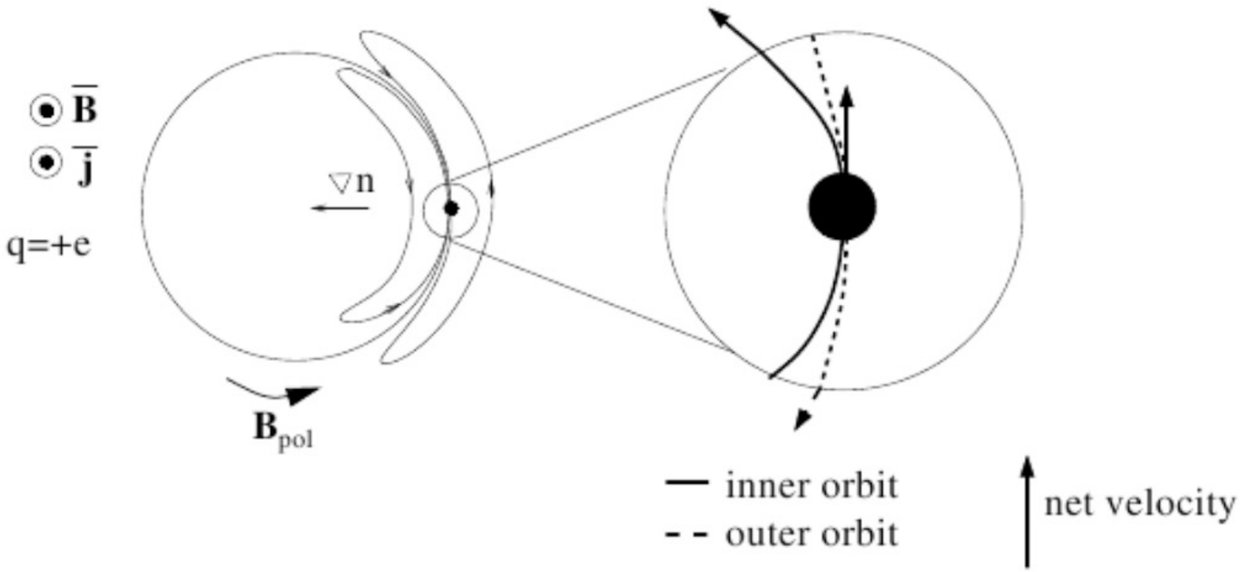
$$\left(v_{||} = \sqrt{2\epsilon} v_{\perp} \approx \sqrt{2\epsilon} v_{th} \right)$$

With: $(n_1 - n_2)/w_B = dn/dr$ current due to trapped particles:

$$j_{gef} \sim \frac{dn}{dr} w_B \epsilon v_{th} \sim \frac{dn}{dr}\sqrt{\epsilon} q r_L v_{th} \sim \frac{dn}{dr}\epsilon^{3/2}\frac{B_\phi}{B_\theta}\frac{\sqrt{kT}}{B} v_{th} \sim \frac{dn}{dr}\frac{T\epsilon^{3/2}}{B_\theta}$$

$$r_B = \frac{r_L q}{\sqrt{\varepsilon}} \qquad q = \varepsilon \frac{B_\phi}{B_\Theta} \qquad r_L \sim \frac{m v_{th}}{B} \sim \frac{1}{B}\sqrt{\frac{k_B T}{m}} \qquad v_{\mathrm{th}} = \sqrt{\frac{k_{\mathrm{B}} T}{m}}.$$

Banana current



Parallel current due to density gradient of trapped particles (for T=const):

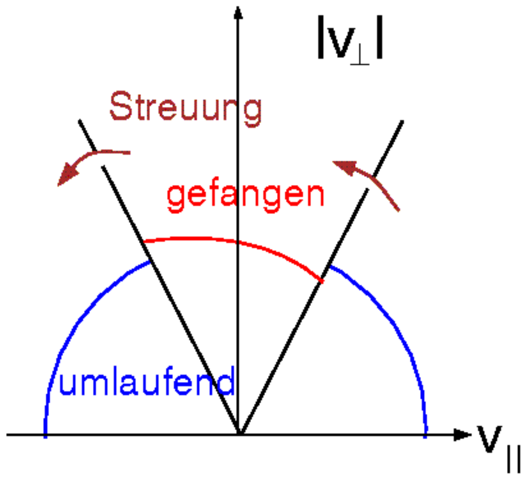
$$J_t \simeq \Delta n v_{||} e. \qquad v_{||} \simeq \sqrt{\epsilon} v_{th} \qquad \Delta n \simeq w_b \frac{d}{dr}\left(\sqrt{\epsilon} n\right)$$

$$J_t \simeq e\, w_b \sqrt{\epsilon} \frac{dn}{dr} \sqrt{\epsilon} v_{th} \qquad w_B = \frac{r_L q}{\sqrt{\varepsilon}} \qquad r_L = \frac{\sqrt{2mkT}}{eB} \qquad v_{\mathrm{th}} = \sqrt{\frac{k_{\mathrm{B}} T}{m}}.$$

$$J_t \sim \frac{q}{B} \sqrt{\epsilon}\, T \frac{dn}{dr} \sim \frac{\epsilon^{3/2}}{B_\theta}\, T \frac{dn}{dr}$$

Banana current also due to temperature gradient

Bootstrap current



Banana current corresponds to shift of the distribution function of the trapped particles

Collisions between trapped and passing particles

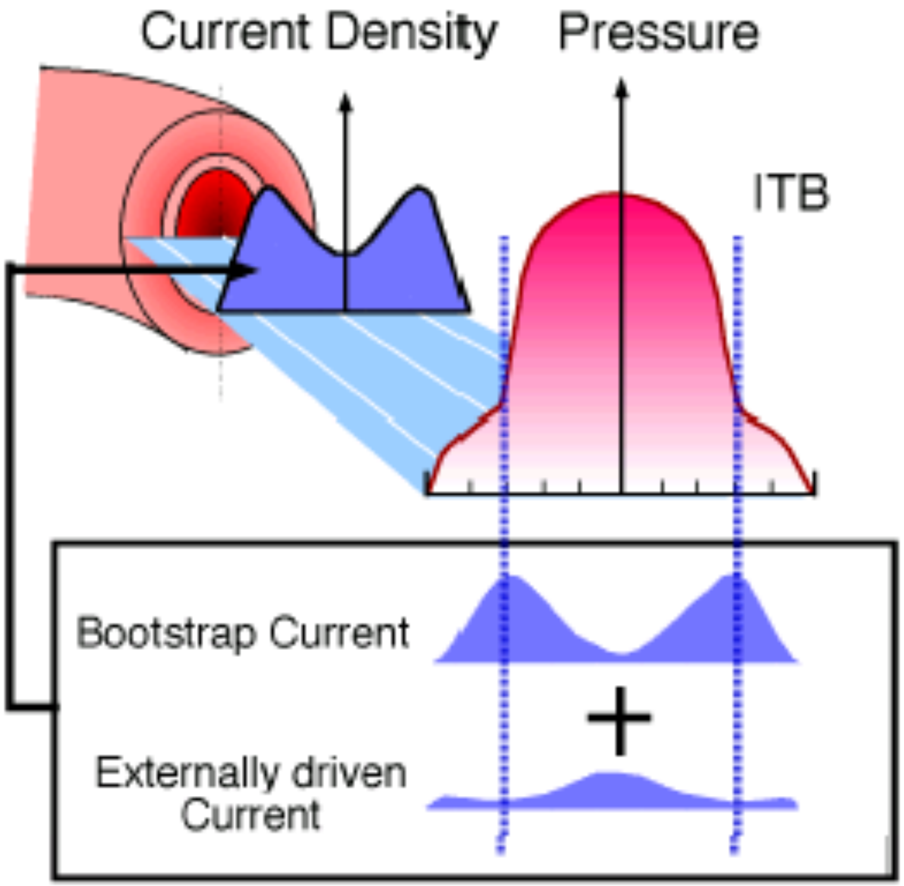
$$F_{gef \rightarrow frei} \sim j_{gef} \nu_{eff} \sim j_{gef} \frac{\nu}{\epsilon}$$

Bootstrap current: $$j_{bs} = j_{gef} + j_{frei} \approx j_{frei} \sim \frac{dn}{dr} \frac{T\sqrt{\epsilon}}{B_\theta}$$ (T=const)

More general: $$j_{bs} \sim \frac{\sqrt{\epsilon}}{B_\theta} \nabla p$$

Detailed calculations show that contribution of ∇n larger than that of ∇T.

Bootstrap current significantly contributes to plasma current (prolonging discharges or even steady state operation)



$$j_{BS} \sim \nabla p$$