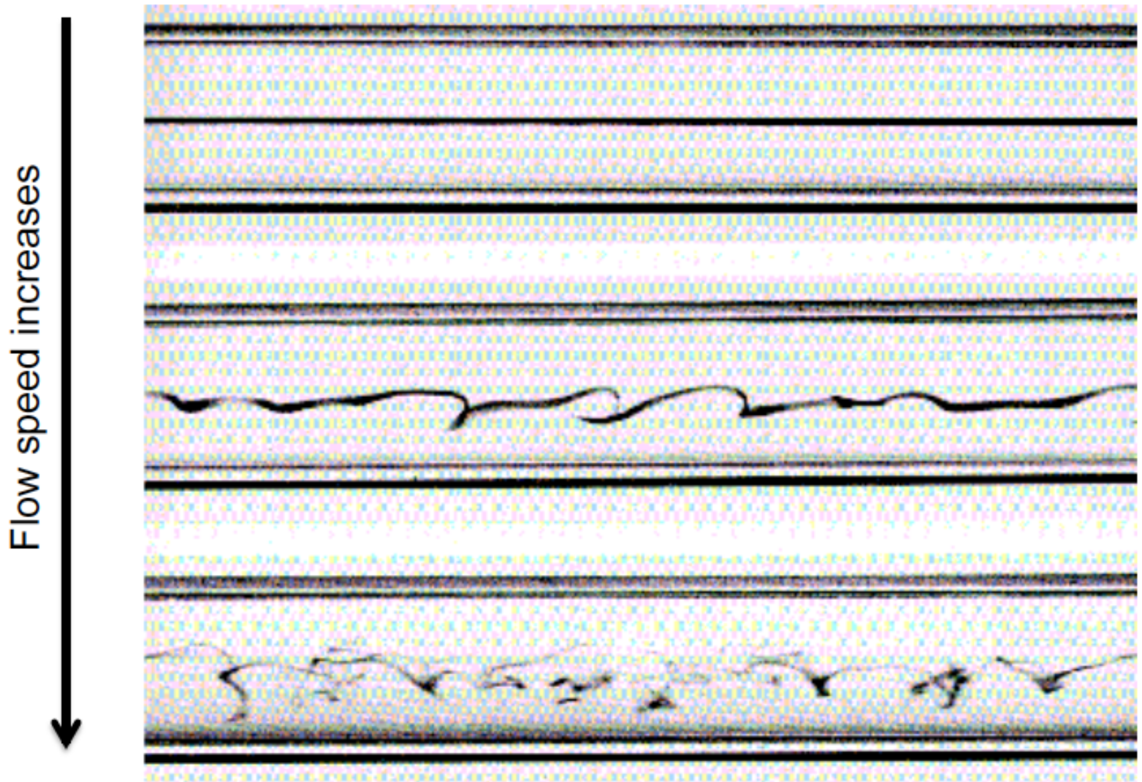
Turbulence

A streaming fluid (e.g. water, hydrodynamic equations) can have non-linear turbulent solutions (Reynolds, 1883)

Osborne Reynolds (1883)

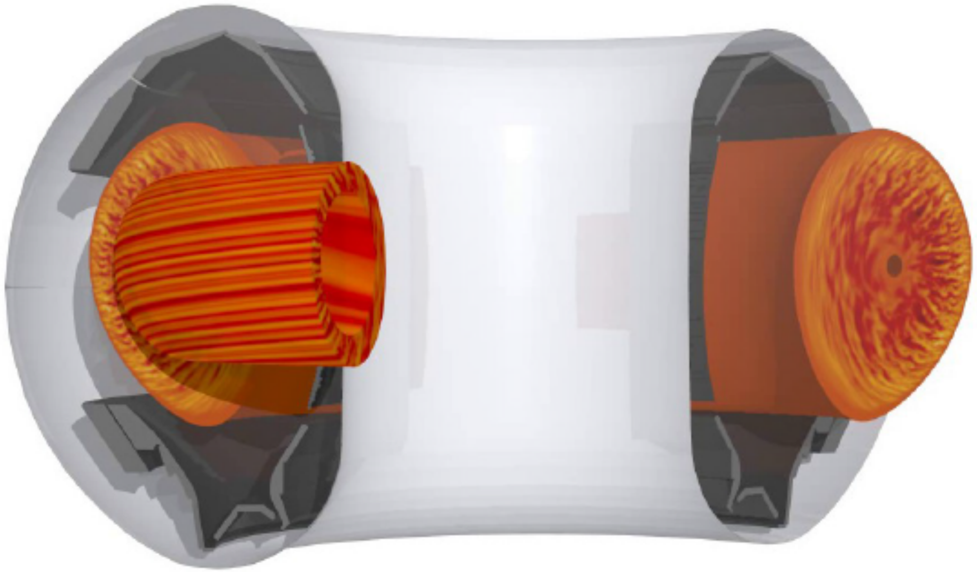


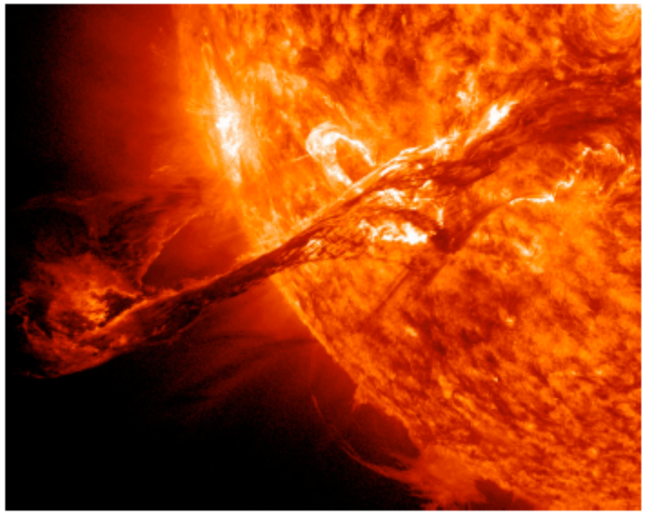
S. Günter with contributions from F. Jenko, U. Stroth, C. Angioni

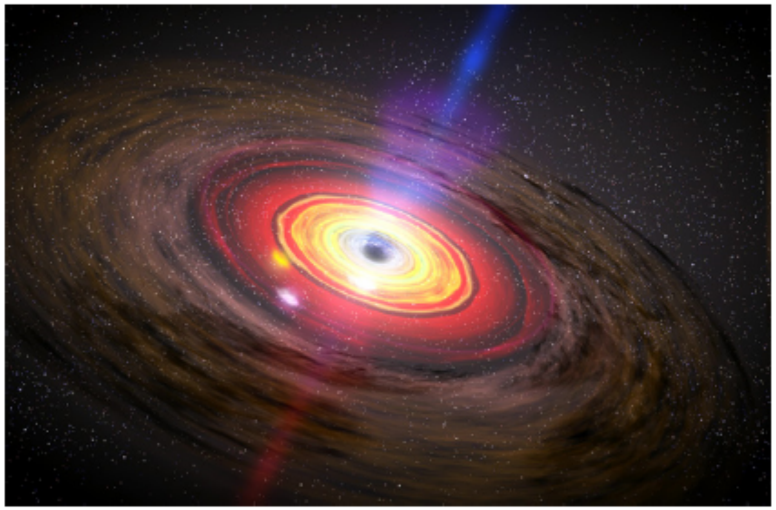
Turbulence – an important problem in fluid dynamics

- Design of airplanes, ships, cars …
- Predict weather and climate
- Blood circulation
- ...

Turbulence in plasmas





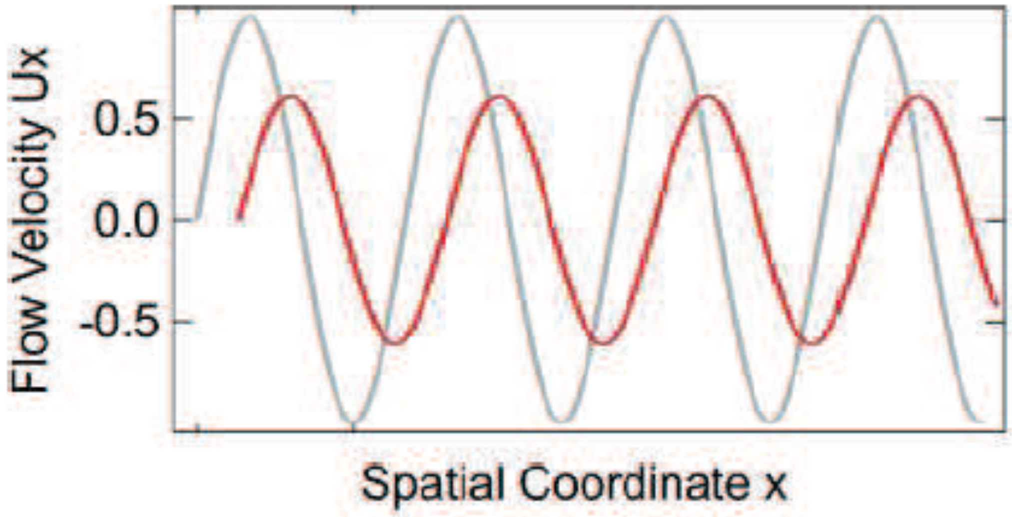


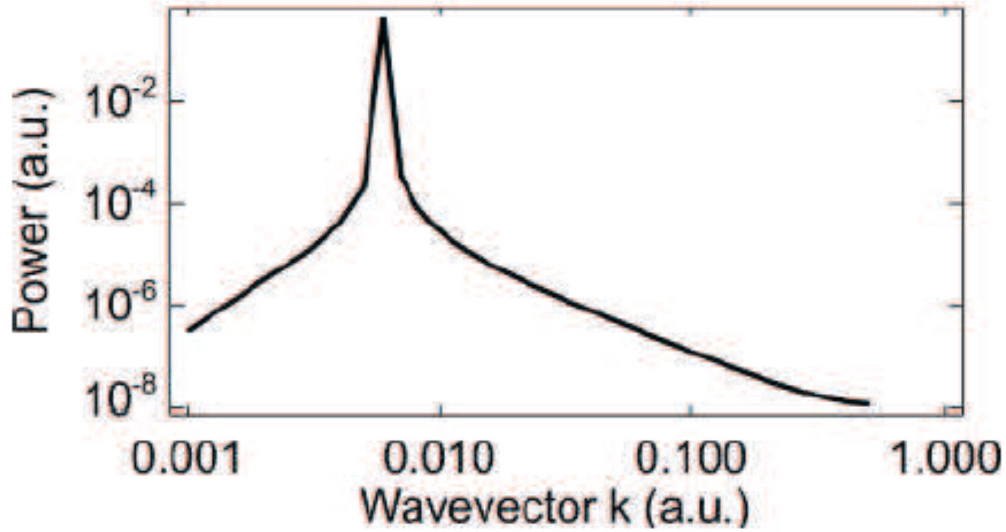
Non-linear physics in fluid equations

Simple linear equation

$$\frac{\partial u}{\partial t} = -u_{ph}\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$

propagation dissipation



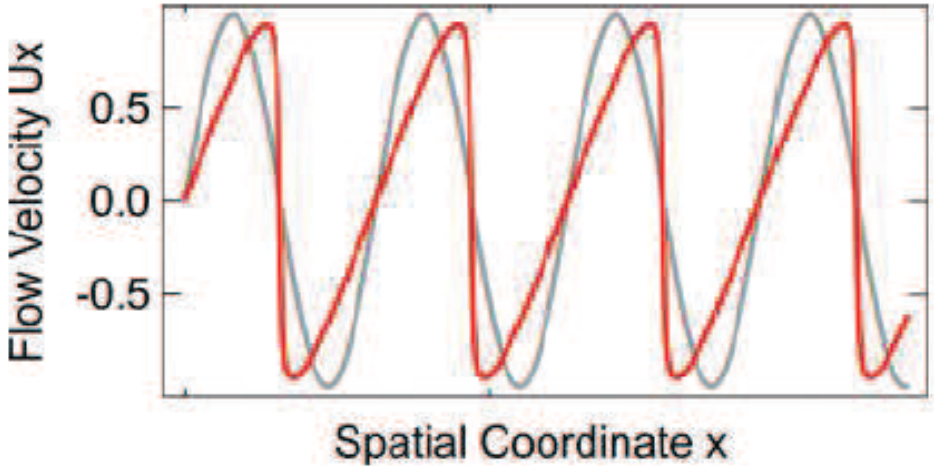


Non-linear physics in fluid equations

Simple linear equation

$$\frac{\partial u}{\partial t} = -u_{ph}\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$

propagation dissipation



Add a nonlinearity

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$

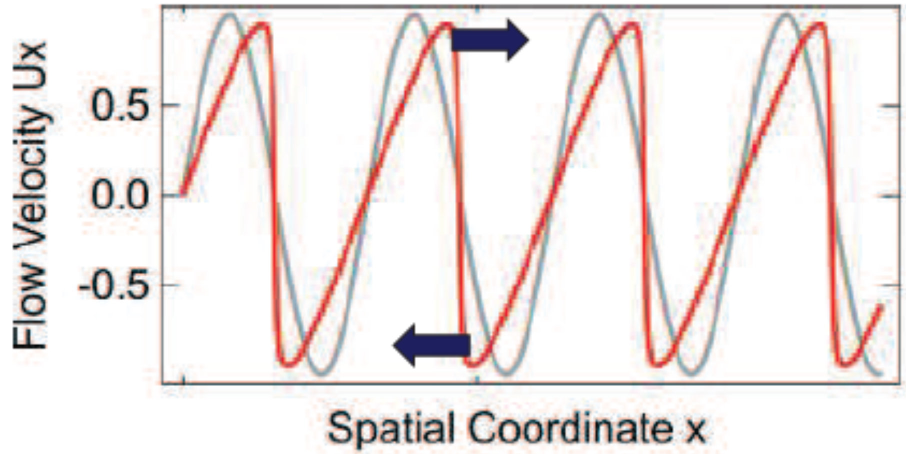
"phase velocity depends on amplitude"

Non-linear physics in fluid equations

Simple linear equation

$$\frac{\partial u}{\partial t} = -u_{ph}\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$

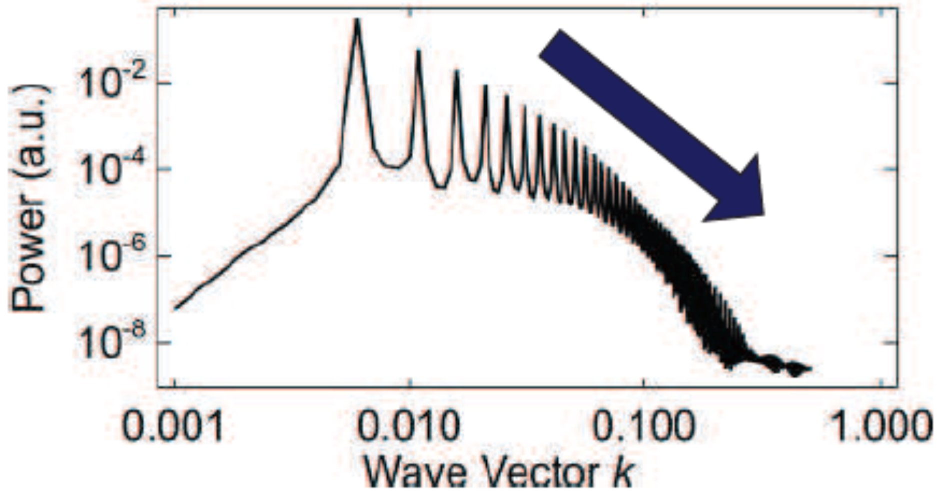
↑ propagation (term $-u_{ph}\frac{\partial u}{\partial x}$) ↑ dissipation (term $\frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$)



Add a nonlinearity

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$

↑ "phase velocity depends on amplitude" (term $-u$)



Nonlinearity creates higher harmonics $\Rightarrow$ **direct cascade**
wave breaking

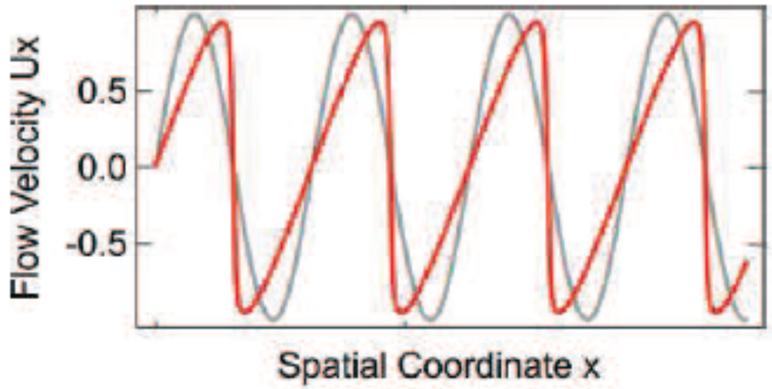
Non-linear physics in fluid equations

Wikipedia

Non-linear physics in fluid equations

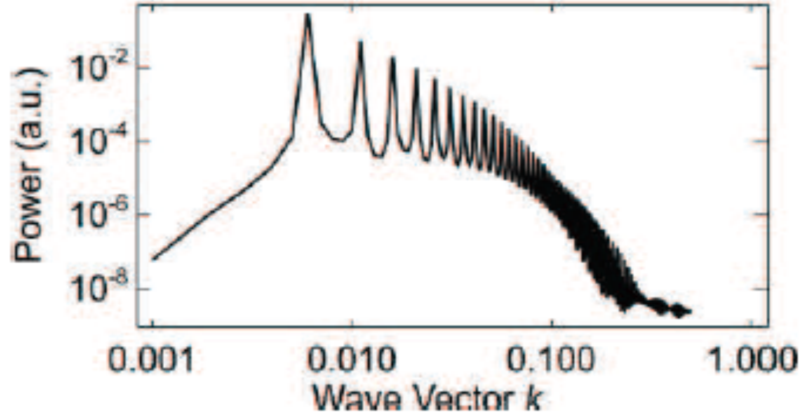
Simple linear equation

$$\frac{\partial u}{\partial t} = -u_{ph}\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$



Add a nonlinearity

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial x^2}$$



3D and add some drive ⇒ Navier-Stokes Equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\hat{p} + \frac{1}{\mathcal{R}_e}\nabla^2\mathbf{u}$$

turbulence drive dissipation

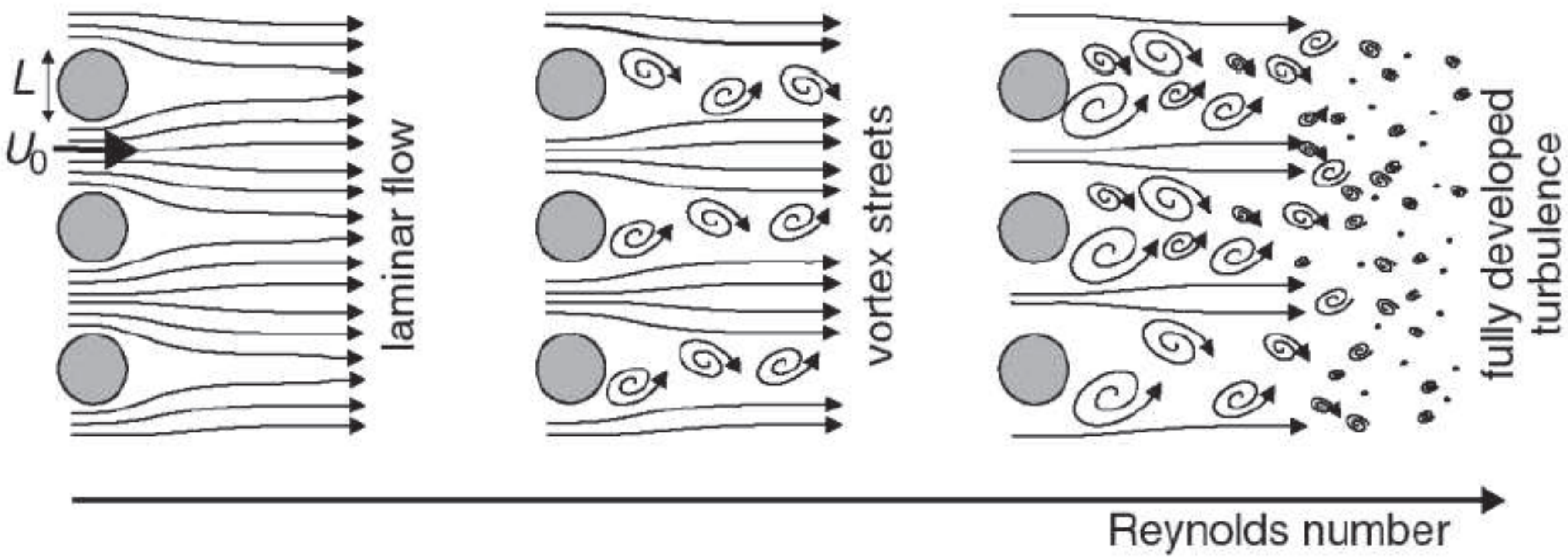
Turbulence in neutral fluids

Navier-Stokes Equation (dimensionless form)

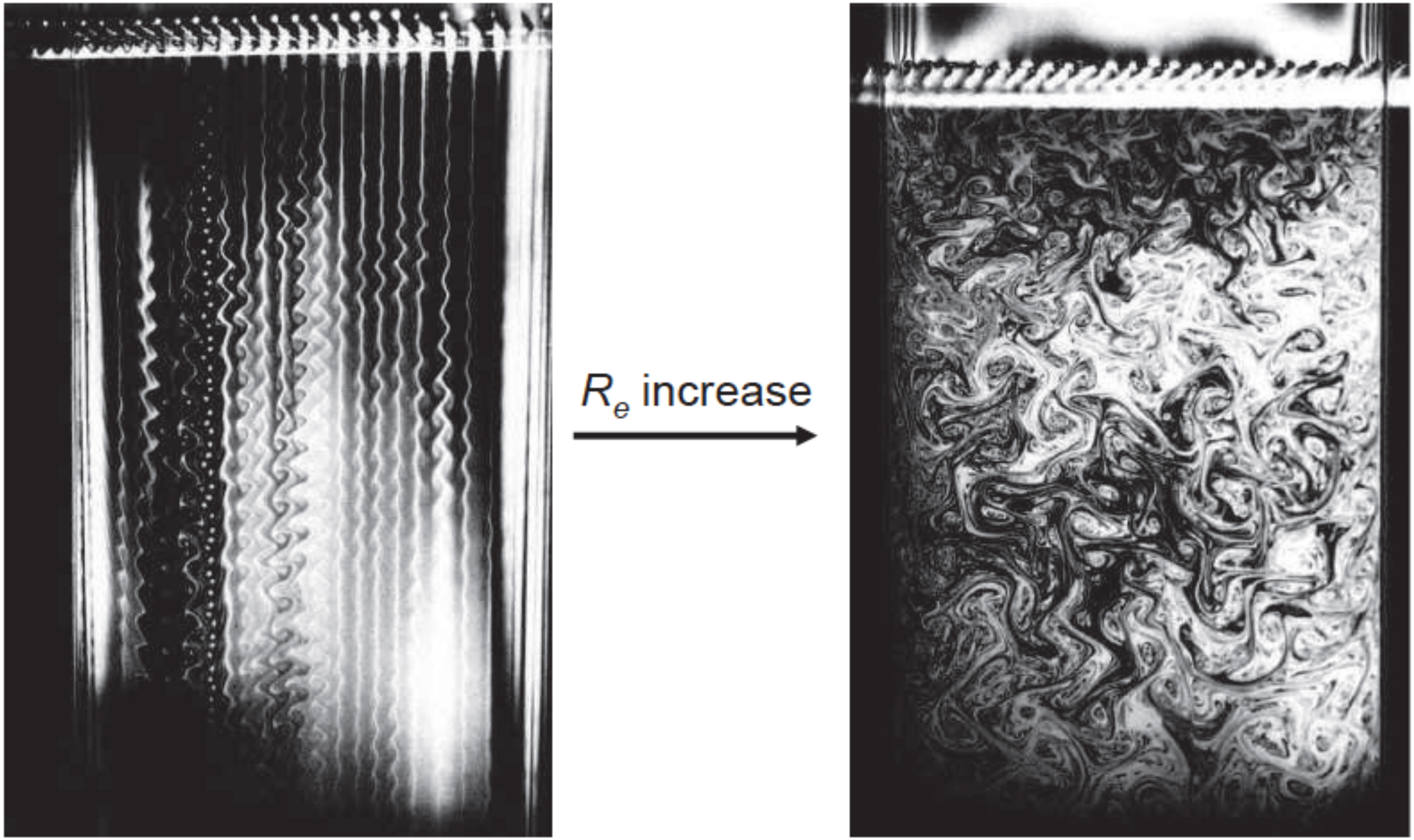
$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}\hat{p} + \frac{1}{\mathcal{R}_e}\nabla^2\mathbf{u}$$

- Dimensionless Reynolds' number decides about the type of flow

$$R_e = \frac{u\nabla u}{\mu\Delta u} = \frac{UL}{\mu}$$



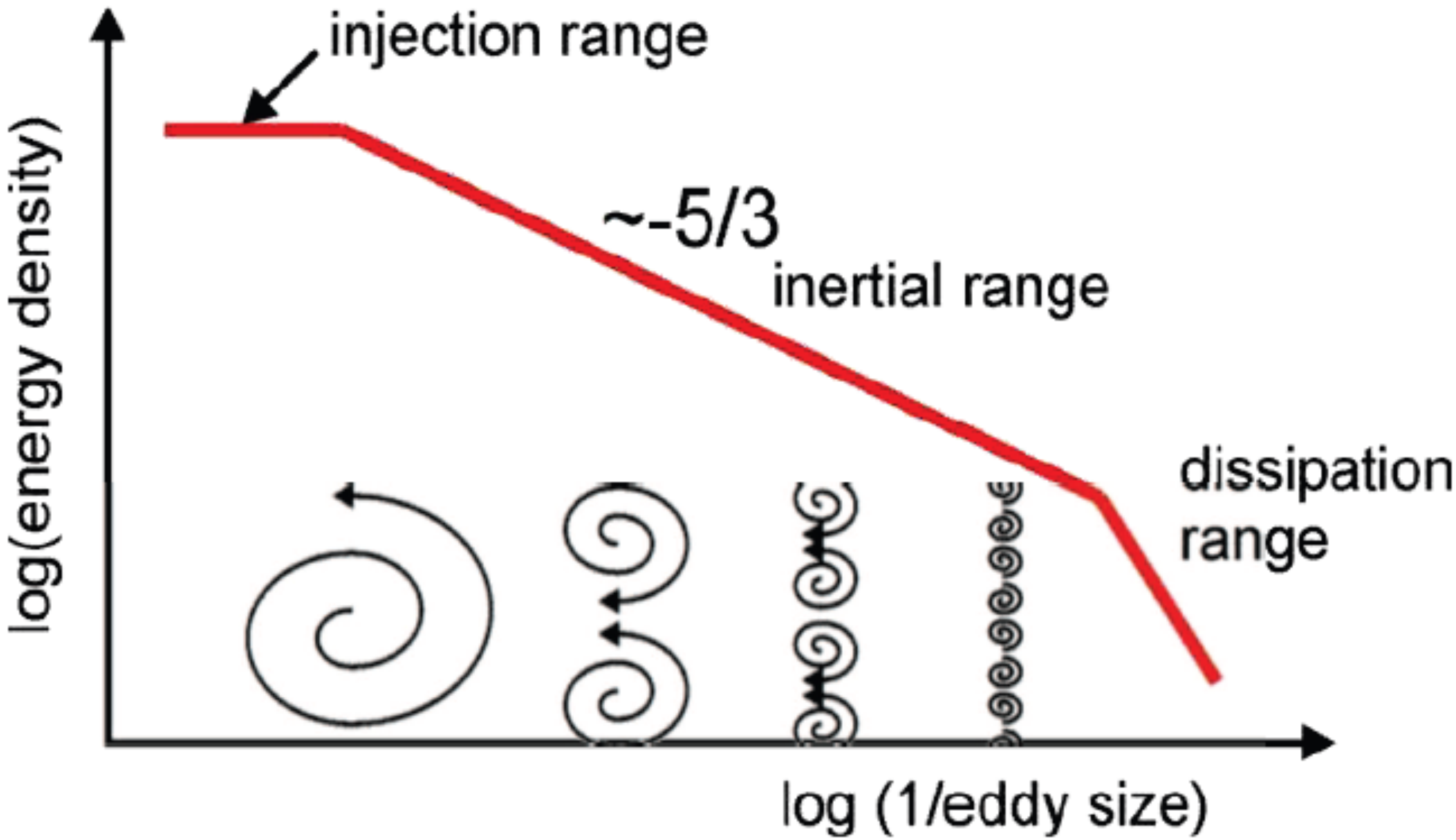
Turbulence in soap films



from maartenrutgers.org/science/turbulence/gallery.html

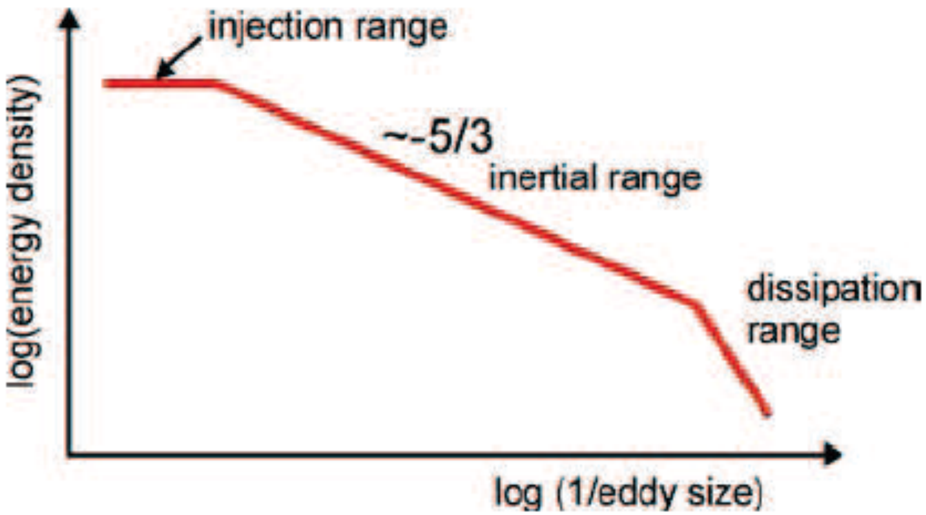
Energy transfer between turbulent scales

K41 Theory, Andrei Kolmogorov 1941:
In 3D turbulence the energy follows a direct cascade

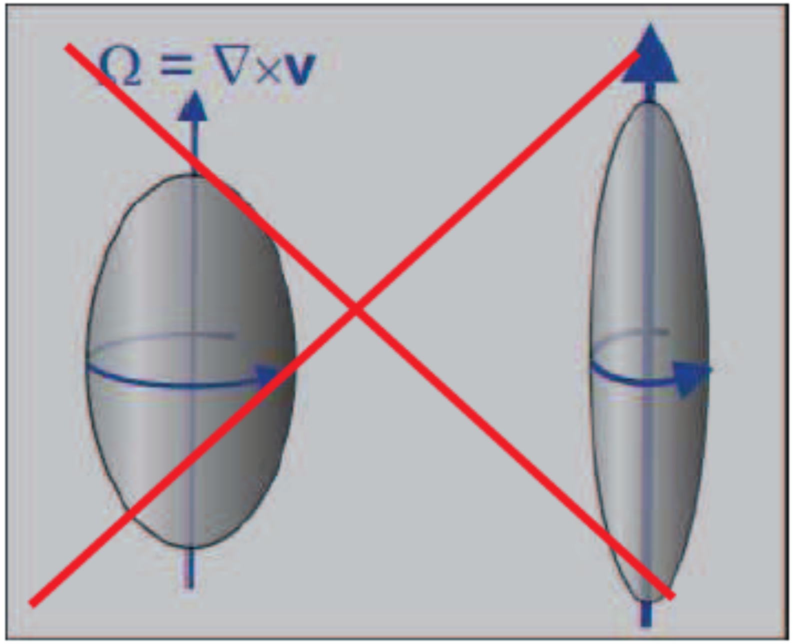


Magnetized plasmas behave like a 2d-fluid

- K41 theory for 3D turbulence



- Eddy stretching as source of vorticity

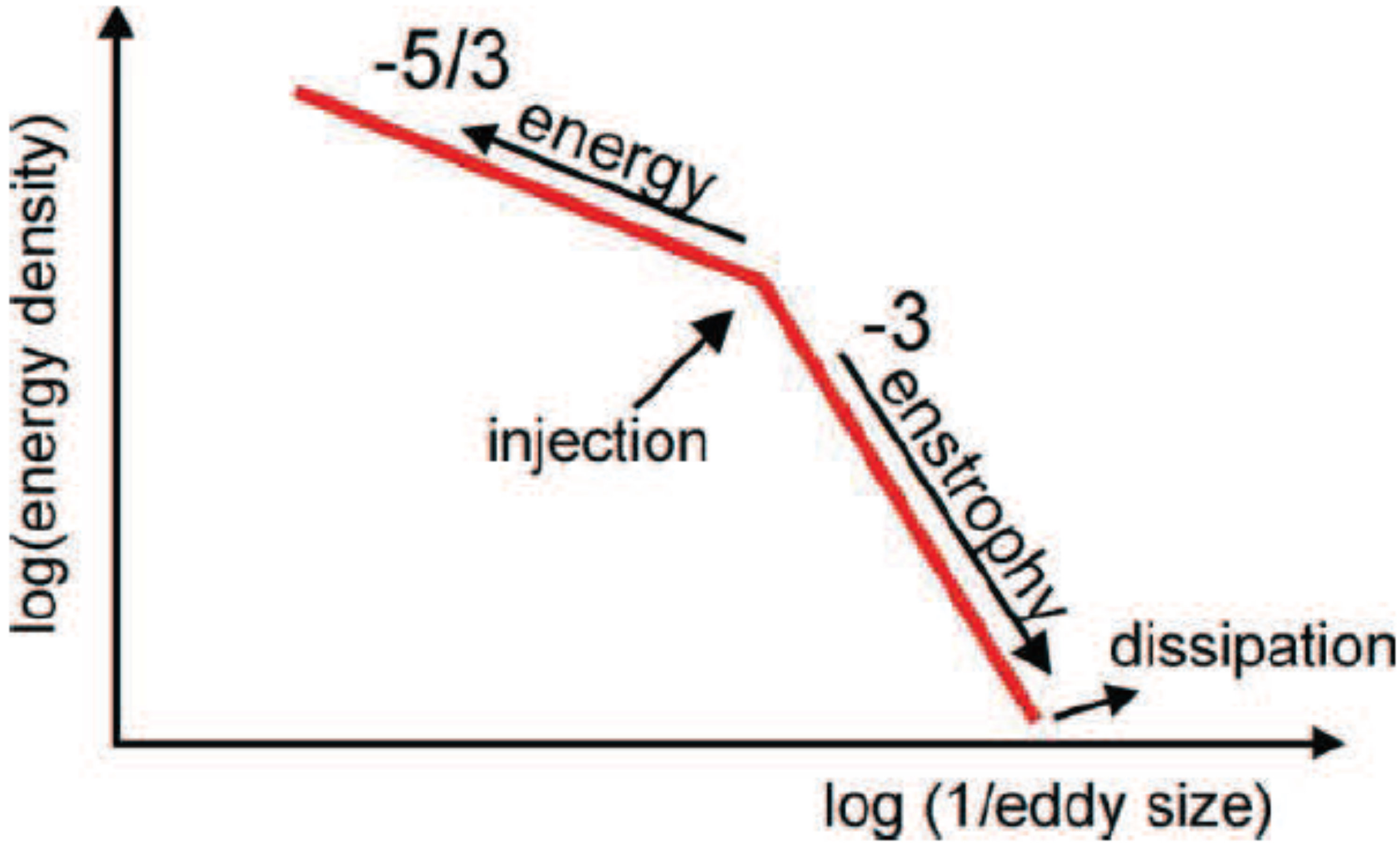


... is not allowed in 2D turbulence

$\Rightarrow$ vorticity is conserved

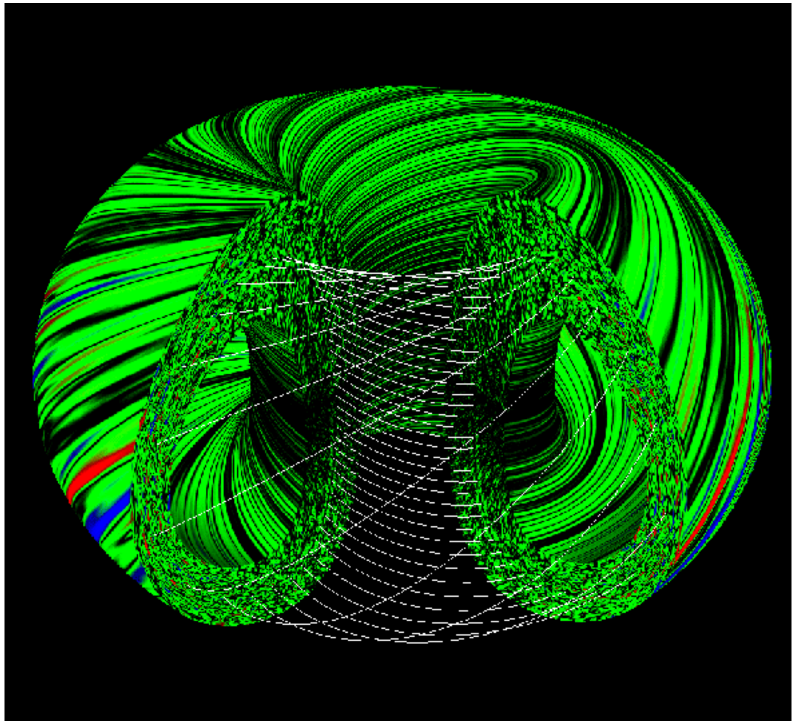
Dual cascade in 2d turbulence

The dual cascade by R. Kraichnan 1967



Energy transferred to large scales ⇒ ”infrared catastrophe”

Fluctuations in a fusion plasma



- Fluctuations in all quantities (n,T,B,...)
- Fluctuations extended along magnetic field lines
- Perpendicular to magnetic field lines size much smaller than system size
- Fluctuation frequency much smaller than gyro-frequency

fluctuations in a plasma

Measured density fluctuations:

$$1\,\mathrm{kHz} \lesssim f \lesssim 100\,\mathrm{kHz} \qquad \tilde{n}_e/n_e \sim 10^{-3} - 10^{-2}$$

What defines the amplitude?

Fluctuations are driven by gradients, thus saturation due to local flattening of gradients

Fluctuation stops for $$\left|\nabla \tilde{n}_e\right| \sim \left|\nabla n_{e0}\right|$$

$$\left|\nabla \tilde{n}_e\right| \sim k_\perp \tilde{n}_e \qquad \left|\nabla n_{e0}\right| \sim n_{e0}/L_n$$

$$L_n \equiv n_e/\left|\nabla n_e\right|$$

$$\boxed{\frac{\tilde{n}_e}{n_{e0}} \sim \frac{1}{k_\perp L_n} = \frac{1}{k_\perp \rho_s}\frac{\rho_s}{L_n}}$$

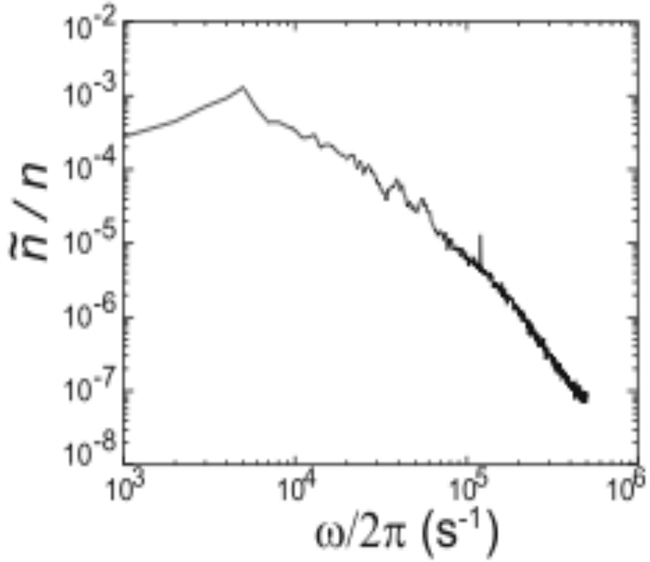
fluctuations in a plasma

Measured density fluctuations:

$$1\,\mathrm{kHz} \lesssim f \lesssim 100\,\mathrm{kHz} \qquad \tilde{n}_e/n_e \sim 10^{-3} - 10^{-2}$$

$$\frac{\tilde{n}_e}{n_{e0}} \sim \frac{1}{k_\perp L_n} = \frac{1}{k_\perp \rho_s}\frac{\rho_s}{L_n}$$

$$k_\perp \rho_s \sim 0.1 - 0.3\,,$$



$$\tilde{n}_e/n_{e0} \sim (3 - 10)\,\rho_s/L_n$$

Density fluctuations particularly strong at plasma edge where L_n small

$$\rho_s \equiv c_s/\Omega_i \;(\mathrm{mit}\; c_s^2 = k_B T_e/m_i)$$

ion gyro-radius with electron temperature

fluctuations in a plasma

Measured density fluctuations:

$$k_\perp \rho_s \sim 0.1 - 0.3$$

- Extremely anisotropic: in parallel direction about $10^3 \ldots 10^4$ times larger extent than in perpendicular direction

- Temperature fluctuations are more difficult to measure, but similar

- B-field fluctuations perpendicular to B, very small parallel to B:

$$\tilde{B}_\perp / B_0 \sim 10^{-5} - 10^{-4}$$

Fluctuation induced transport

Radial particle transport due to fluctuating electric and magnetic fields:

$$\tilde{v}_r = \tilde{v}_{Er} + v_{||} \tilde{B}_r / B_0$$

Velocity distribution function for electrons: perturbed Maxwell-distribution:

$$f_{e0} = n_{e0} \left(\frac{m_e}{2\pi k_B T_{e0}} \right)^{3/2} \exp\left[-\frac{mv^2}{2 k_B T_{e0}} \right] \qquad f_e = f_{e0} + \tilde{f}_e , \quad \tilde{f}_e \ll f_{e0}$$

Particle transport:

$$\Gamma \equiv \int \tilde{v}_r \tilde{f}_e d^3v = \tilde{v}_{Er} \int \tilde{f}_e d^3v + (\tilde{B}_r / B_0) \int v_{||} \tilde{f}_e d^3v$$

Moments of perturbed distribution function (linearised):

$$\int \tilde{f}_e d^3v = \tilde{n}_e \qquad \int v_{||} \tilde{f}_e d^3v = n_{e0} \tilde{u}_{e||}$$

Fluctuation induced transport

Moments of perturbed distribution function (linearised):

$$\int \tilde{f}_e d^3v = \tilde{n}_e \qquad \int v_{||} \tilde{f}_e d^3v = n_{e0}\, \tilde{u}_{e||}$$

Finite particle transport due to fluctuating fields only if there is a phase relation between density and velocity fluctuations (time and space averaging<>):

$$\langle \Gamma \rangle = \langle \tilde{n}_e \tilde{v}_{Er} \rangle + n_{e0} \langle \tilde{u}_{e||} \tilde{B}_r \rangle / B_0$$

$$\langle \tilde{n}_e \rangle = \langle \tilde{u}_{e||} \rangle = \langle \tilde{v}_{Er} \rangle = \langle \tilde{B}_r \rangle = 0$$

Heat flux (due to electrons):

$$\langle Q_e \rangle = \frac{3}{2} k_B T_{e0} \langle \Gamma \rangle + \frac{3}{2} n_{e0} k_B \langle \tilde{T}_e \tilde{v}_{Er} \rangle + \langle \tilde{q}_{e||} \tilde{B}_r \rangle / B_0 + p_{e0} \langle \tilde{u}_{e||} \tilde{B}_r \rangle / B_0$$

Drift waves

Homogeneous magnetic field in z-direction: $\mathbf{B}=B\mathbf{e}_z$

Force balance for electrons: $$\nabla p_e + en_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

Small B-field perturbation, static equilibrium, ideal plasma:

Electrostatic fluctuations $$\mathbf{E} = -\nabla\tilde{\phi}$$

Small density fluctuations $$n_e = n_{e0} + \tilde{n}_e,\ \tilde{n}_e \ll n_{e0}$$

Slowly varying background profiles:

$$p_e = n_e k_B T_{e0} \qquad \nabla n_{e0} = -(n_{e0}/L_n)\,\hat{\mathbf{x}}$$

Drift waves

Parallel component of force balance: $\nabla_{\parallel} p_e = -e n_e E_{\parallel}$

No equilibrium pressure gradient along magnetic field lines:

$$\nabla_{\parallel} p_e = \nabla_{\parallel} \tilde{p}_e = T_{e,0} \nabla_{\parallel} \tilde{n}_e$$

Isothermal (no temperature gradient along field lines)

$$-e n_{e,0} E_{\parallel} = e n_{e,0} \nabla_{\parallel} \tilde{\phi}$$

Boltzmann-relation: Electron density perturbation leads to potential perturbation (no phase delay!)

$$T_{e,0} \nabla_{\parallel} \tilde{n}_e = e n_{e,0} \nabla_{\parallel} \tilde{\phi}$$

$$\boxed{\frac{\tilde{n}_e}{n_{e0}} = \frac{e\tilde{\phi}}{T_{e0}}}$$

Drift waves

Force balance for electrons: $\nabla p_e + e n_e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \quad \Big| \times \vec{B}/(e n_e B^2)$

Perpendicular component of force balance:

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\nabla p_e \times \mathbf{B}}{e n_{e0} B^2} \equiv \mathbf{v}_E + \mathbf{v}_{de}$$

Linearised continuity equation (static equilbrium: v_0=0): $\partial_t \tilde{n}_e + \nabla \cdot (n_{e0} \mathbf{v}_\perp) = 0$

$$\nabla \cdot (n_{e0} \mathbf{v}_{de}) \propto \nabla \cdot (\nabla p_e \times \hat{\mathbf{z}}) = 0 \qquad \nabla \cdot \mathbf{v}_E \propto \nabla \cdot (\nabla \tilde{\phi} \times \hat{\mathbf{z}}) = 0$$

$$\boxed{\partial_t \tilde{n}_e + \mathbf{v}_E \cdot \nabla n_{e0} = \partial_t \tilde{n}_e - (n_{e0}/L_n)\, v_{Ex} = 0}$$

Drift waves

$$\partial_t \tilde{n}_e + \mathbf{v}_E \cdot \nabla n_{e0} = \partial_t \tilde{n}_e - (n_{e0}/L_n)\, v_{Ex} = 0$$

Ansatz for perturbation: $\tilde{n}_e \sim \tilde{\phi} \sim \exp\left(i\vec{k}\vec{y} - i\omega t\right)$

$$-i\omega\tilde{n}_e = \frac{n_{e,0}}{L_n} v_{Ex} \qquad v_{Ex} = \frac{\vec{E}\times\vec{B}}{B^2} = \frac{-\nabla\phi_y}{B}$$

$$-i\omega\tilde{n}_e = -i\frac{n_{e,0}}{L_n}\frac{k_y\tilde{\phi}_y}{B} \qquad \frac{\tilde{n}_e}{n_{e0}} = \frac{e\tilde{\phi}}{T_{e0}}$$

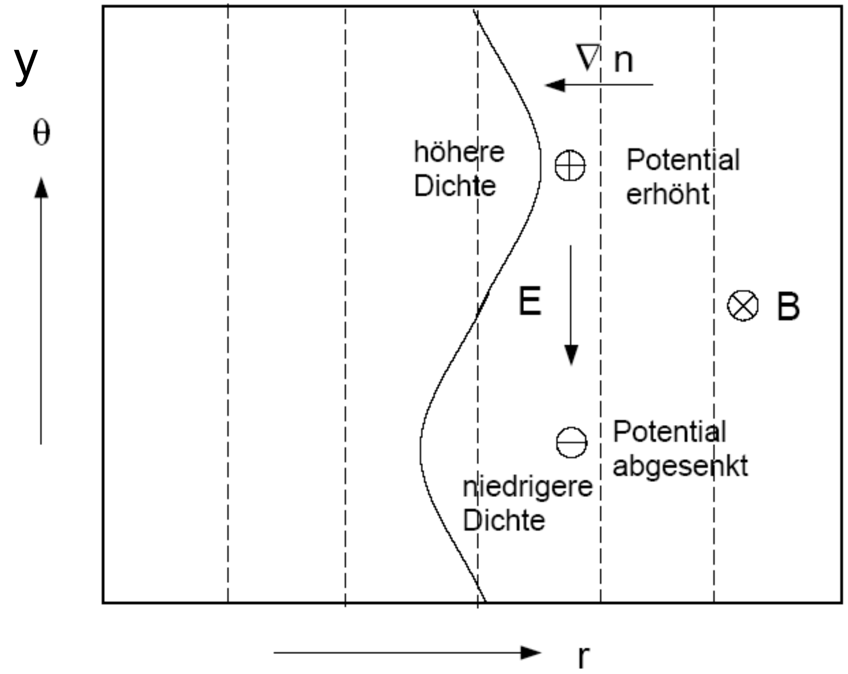
$$\omega_D = \frac{k_y T_{e0}}{eBL_n}$$

Phase velocity: $\frac{\omega_D}{k_y} = \frac{\partial\omega_D}{\partial k_y} = |v_{de}|$

Drift waves

$$\partial_t \tilde{n}_e + \mathbf{v}_E \cdot \nabla n_{e0} = \partial_t \tilde{n}_e - (n_{e0}/L_n)\, v_{Ex} = 0$$

$$\omega_D = \frac{k_y T_{e0}}{eBL_n} \qquad \frac{\omega_D}{k_y} = \frac{\partial \omega_D}{\partial k_y} = |v_{de}|$$



Propagation in y direction

Drift waves in an ideal plasma are marginally stable (no damping, no instability)

With collisions (or Landau damping) (delayed parallel response):

$$\frac{\tilde{n}_e}{n_{e0}} = \frac{e\tilde{\phi}}{T_{e0}}(1 - i\delta)$$

-> complex frequency, drift waves are unstable

$$\omega_D \rightarrow \frac{\omega_D}{1 - i\delta} \approx \omega_D (1 + i\delta)$$

Drift waves grow linearly, until non-linearies play important role:

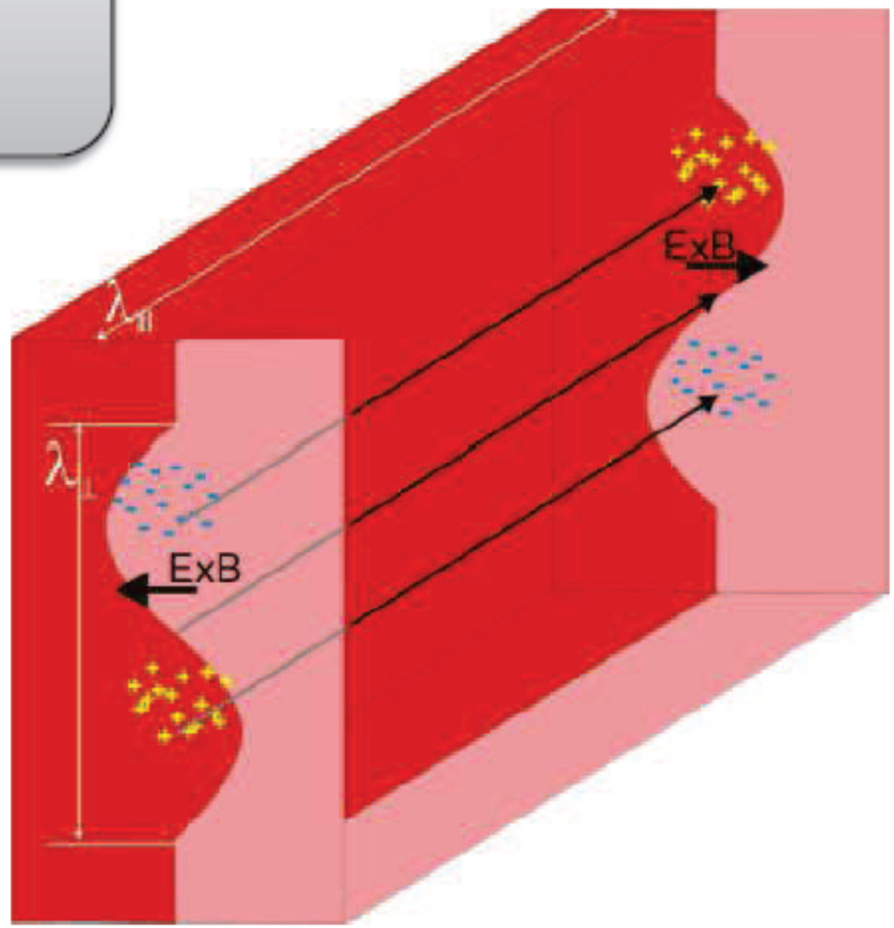
$$\partial_t \tilde{n}_e + \mathbf{v}_E \cdot \tilde{n}_e = \ldots$$

Numerical treatment – large scale turbulece codes

Drift wave instability

Perturbations are

- with finite parallel wave length
- phase(n,ϕ) = 0
- destabilised by resistivity



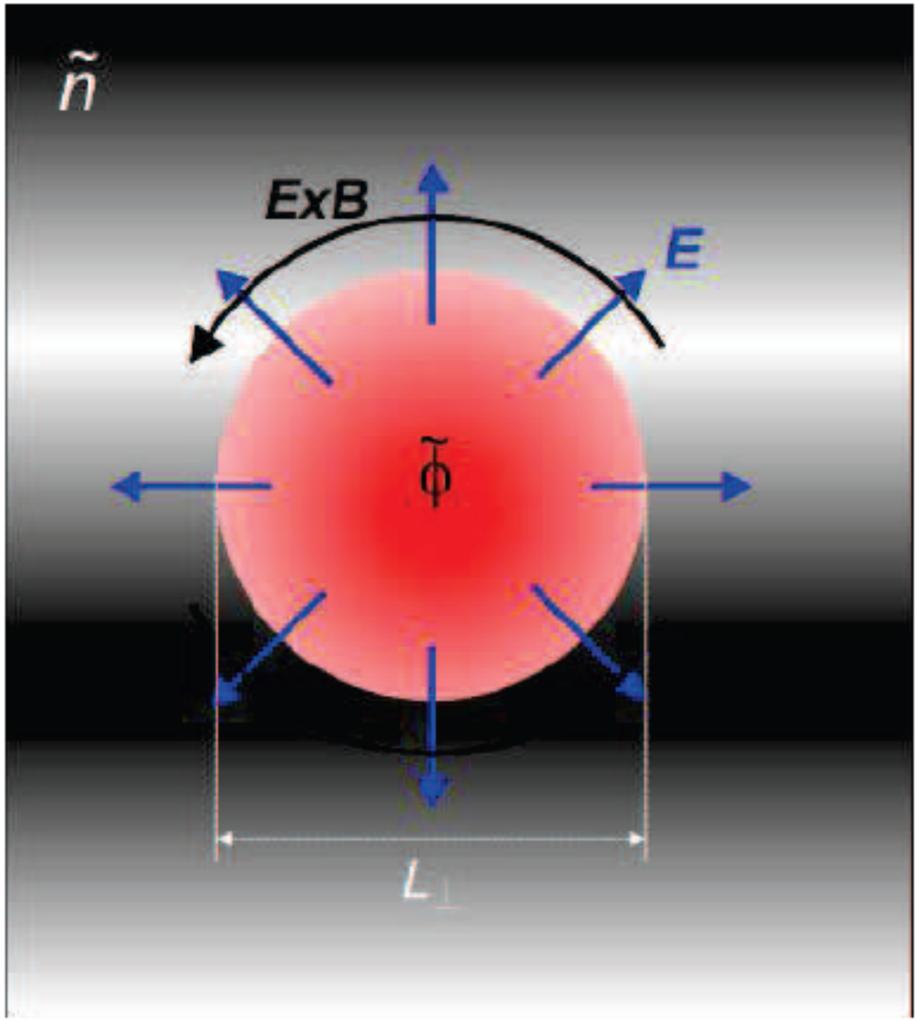
Electrostatic transport

Diffusion from random walk:

- step size: $L_\perp = 2\pi/k_\perp$
- linear growth rate γ
- step time: $\tau = 1/\gamma$

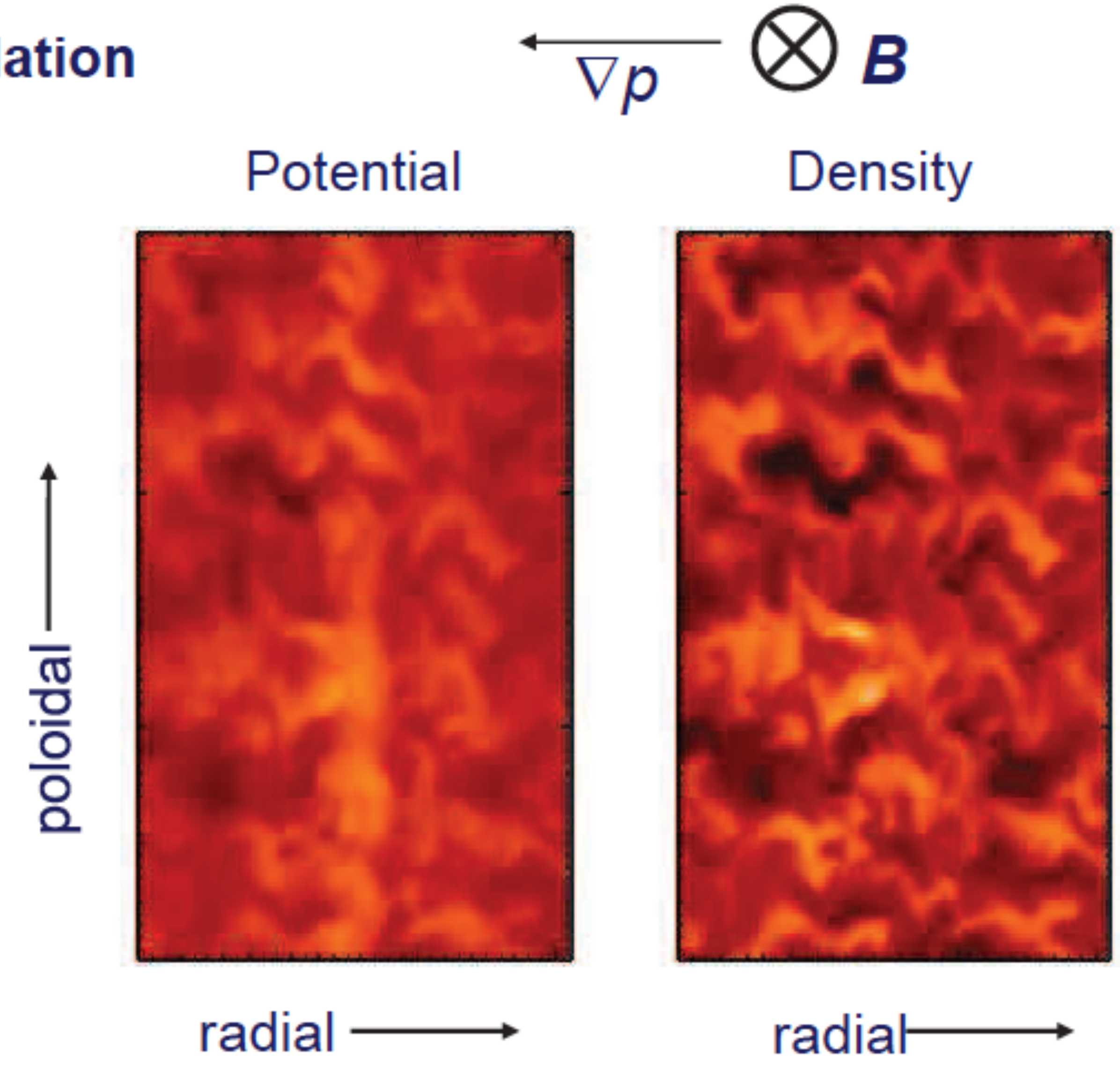
Mixing length estimate of the diffusion coefficient:

$$D \sim \frac{\gamma}{k_\perp^2}$$

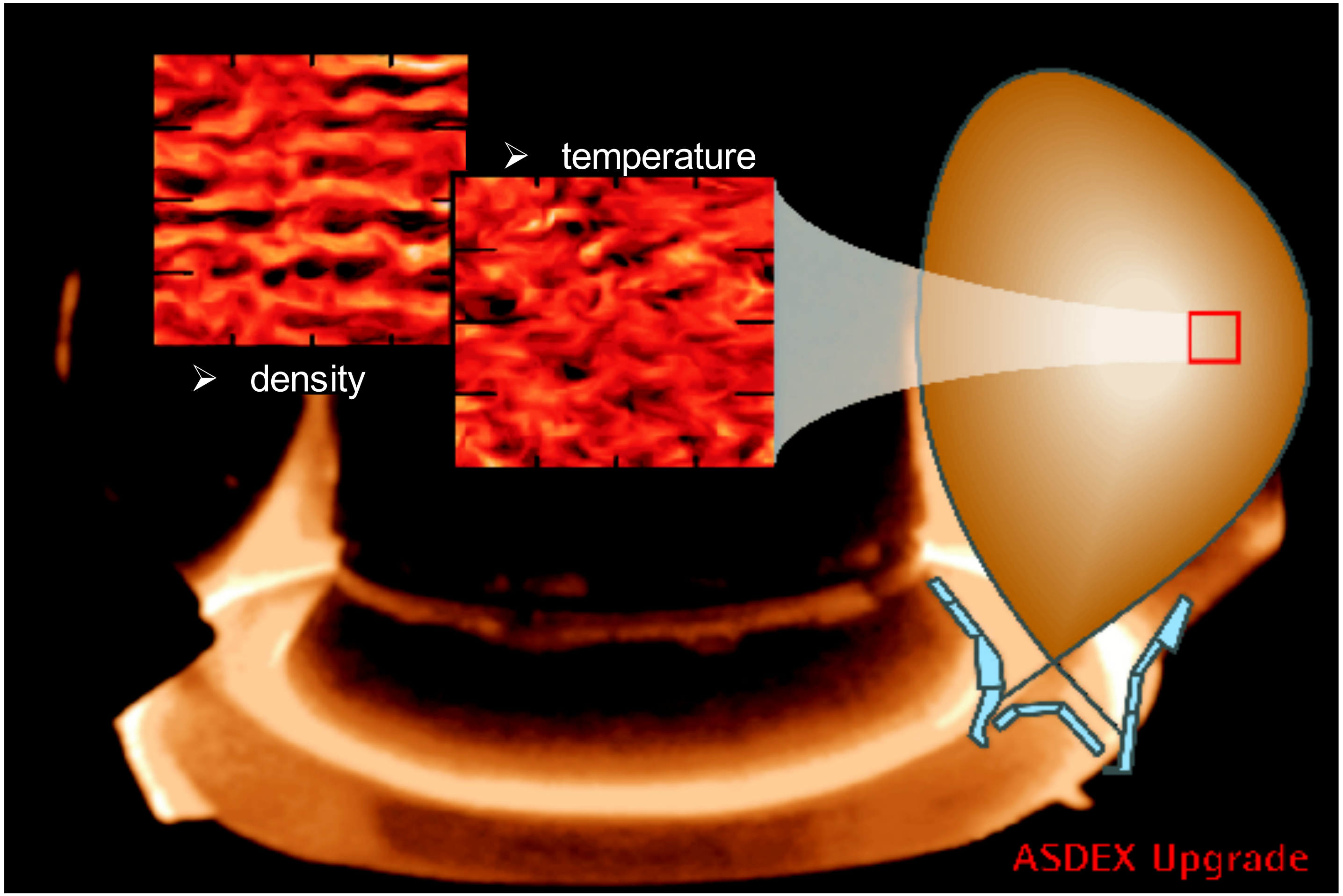


Fully developed turbulence

DALF3 simulation



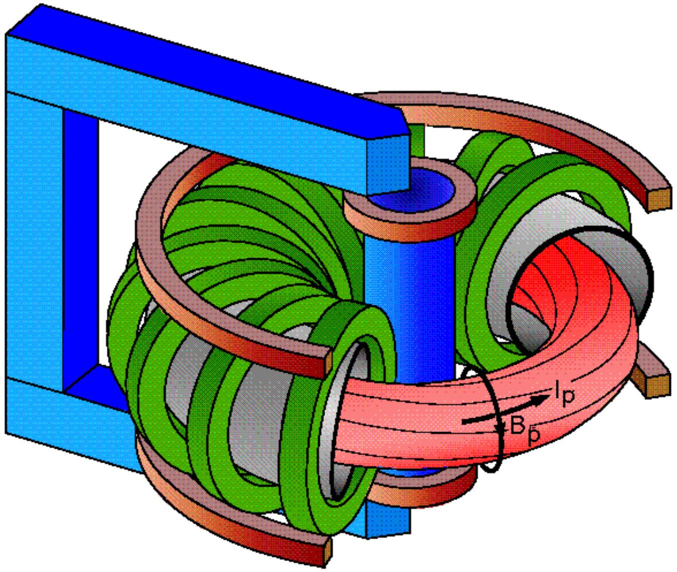
Turbulence simulations for ASDEX Upgrade



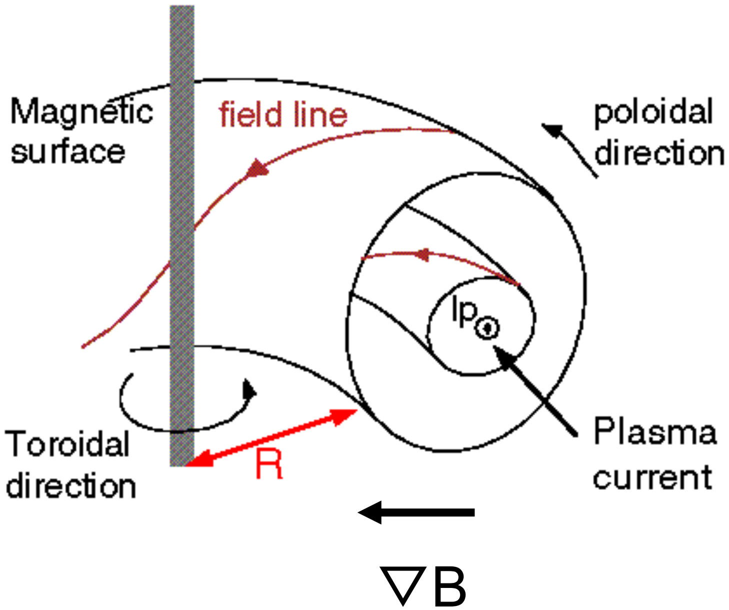
• radial extension of eddies: 1 - 2 cm
• typical life time: 0.5 - 1 ms

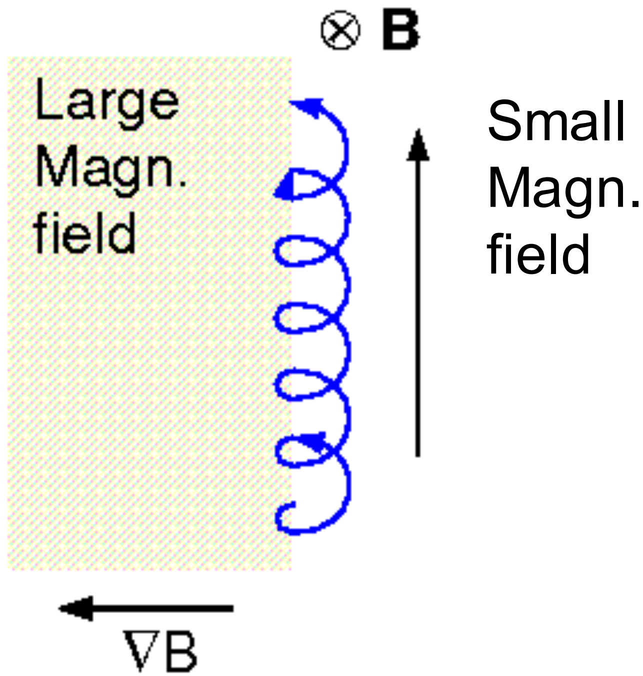
➢ Anomalous transport-coefficients are of order of the measured ones: ~1 m^2/s

Drift in inhomogenous magnetic fields



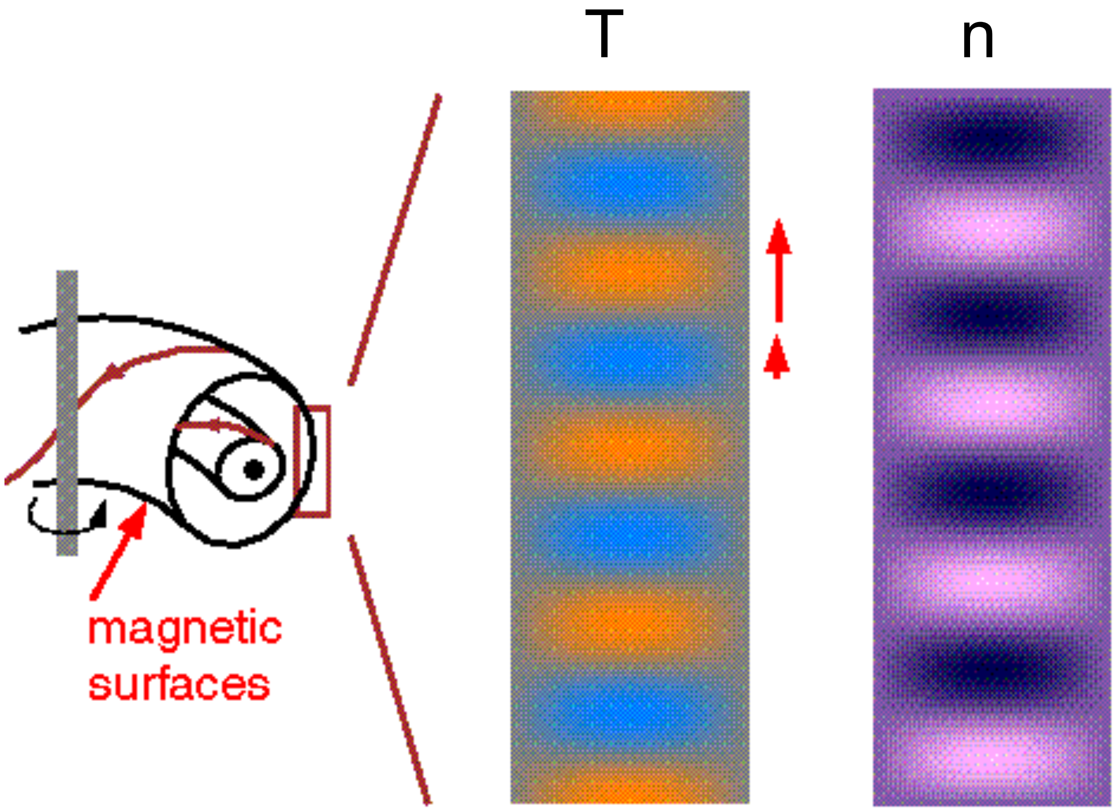
Magnetic field in toroidal geometry is inhomogeneous





Example for a mode that leads to turbulence in a Tokamak

Drift in inhomogeneous magnetic field is temperature dependend



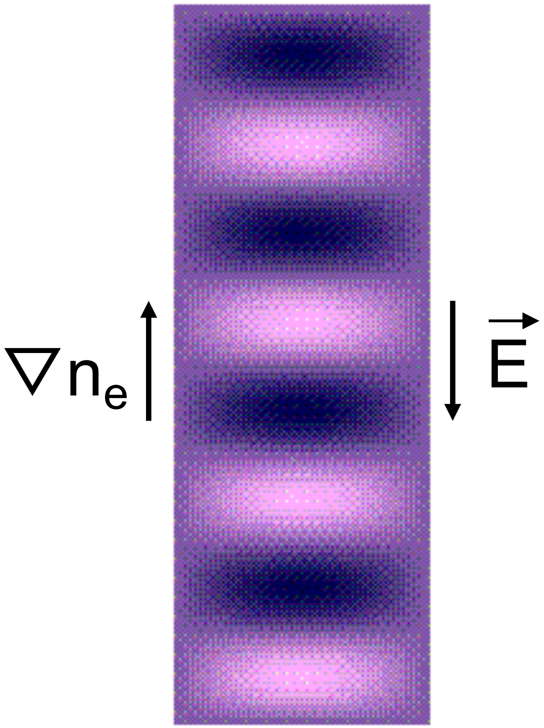
Drift velocity increases with temperature

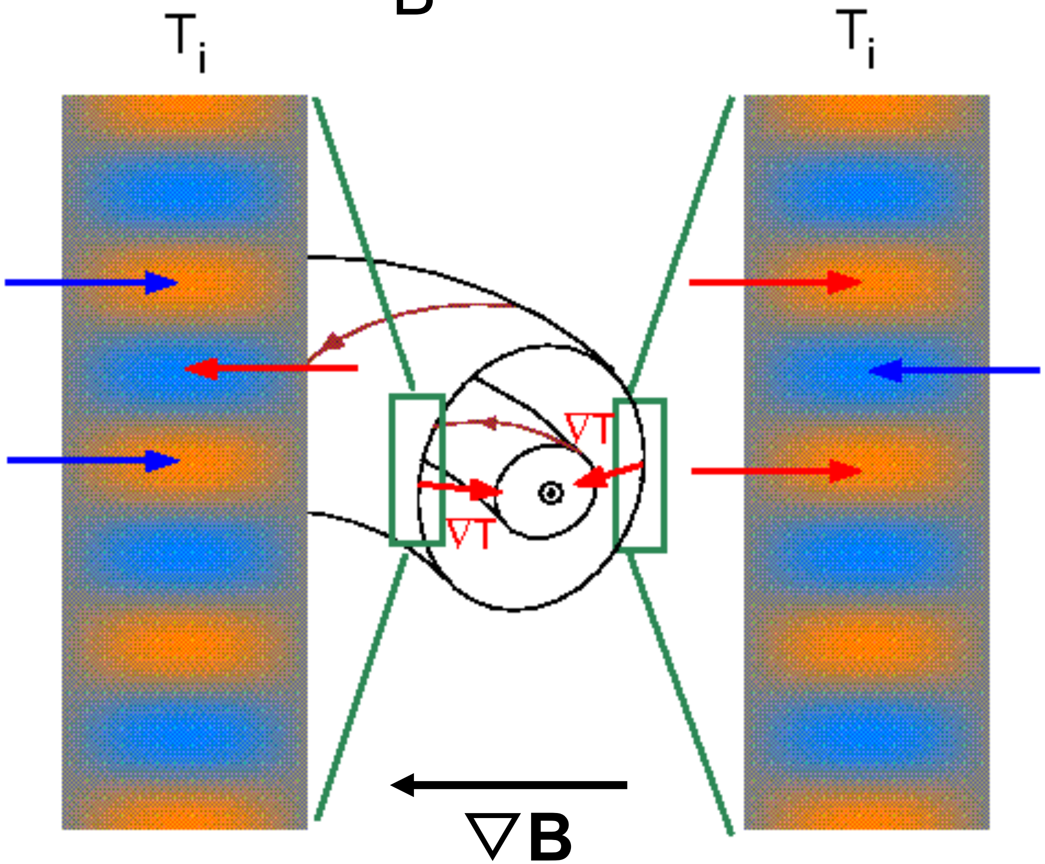
Initial temperature perturbation leads to density perturbation (90° phase shifted)

ITG mode (ion temperature gradient mode)

$$\vec{E} = -\frac{T\,\nabla n_e}{e\,n_e}$$

$$\vec{v}_E = -\frac{1}{B^2}\vec{B}\times\vec{E}$$





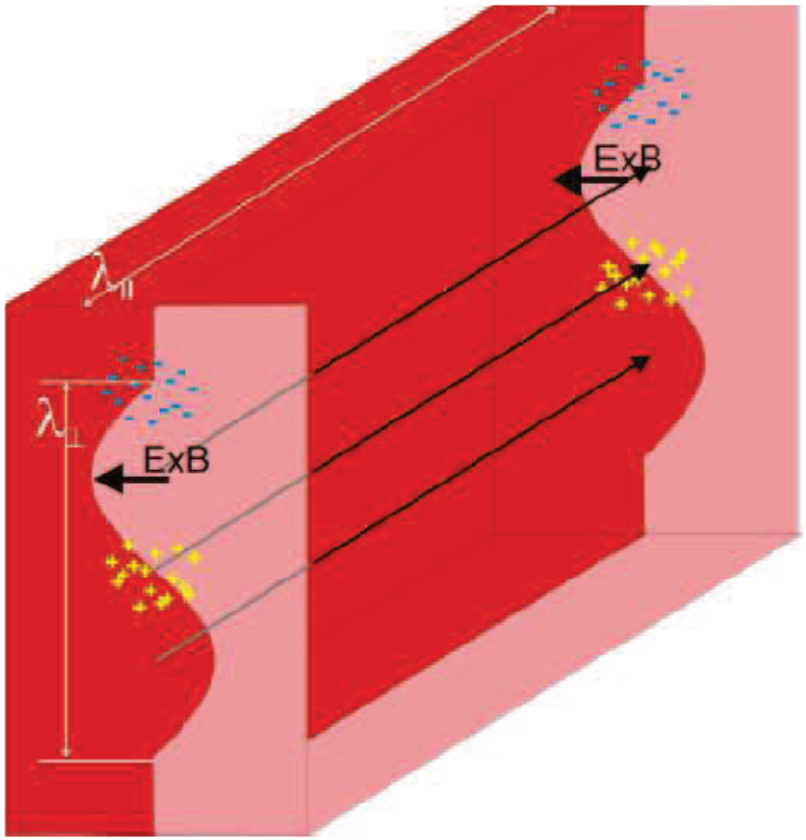
density perturbation causes potential perturbation

Resulting ExB drift amplifies initial perturbation at low B-field side

ITG mode (ion temperature gradient mode)

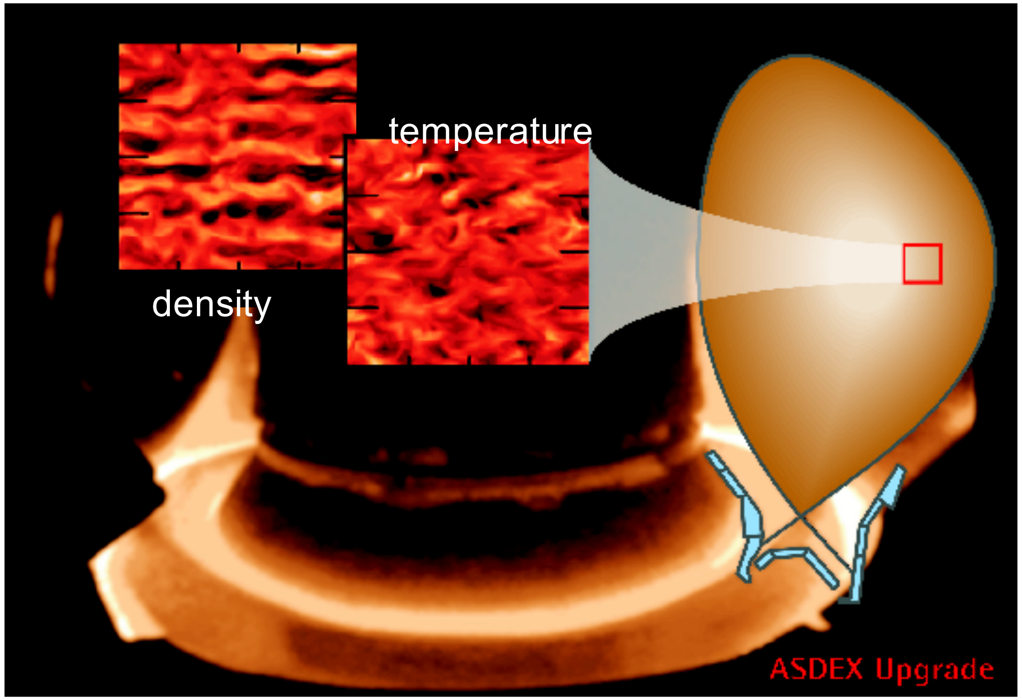
Perturbations are

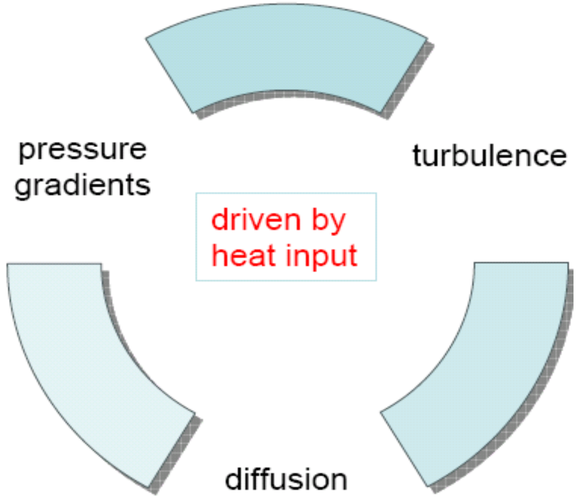
- constant on field linie
- with cross-phase(n,ϕ) = $\pi/2$
- destabilised by curvature



Understanding of the turbulent transport

Ab-initio physics models ⟺ High performance computing



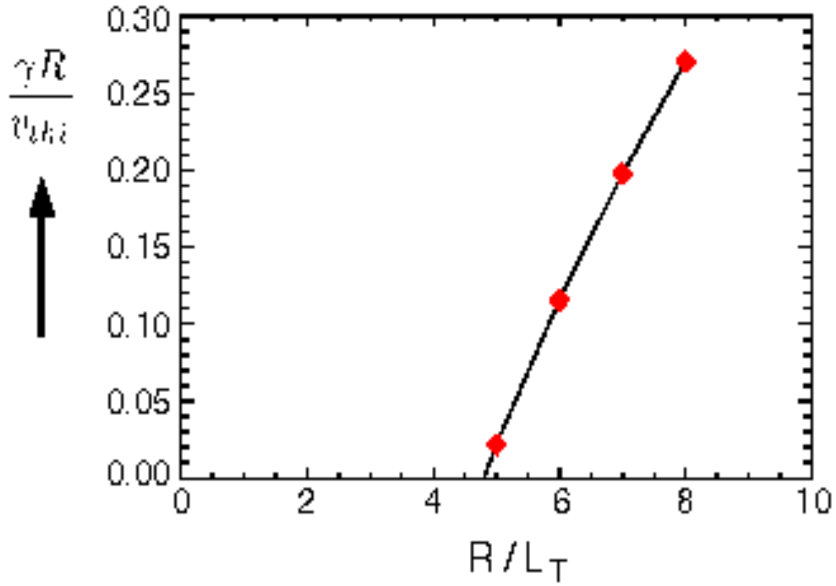


- geometry important
- extremly anisotrop
- kinetic effects

ITG causes 'stiff' temperature profiles

Critical temperature gradient:
Mode grows exponentially above this threshold

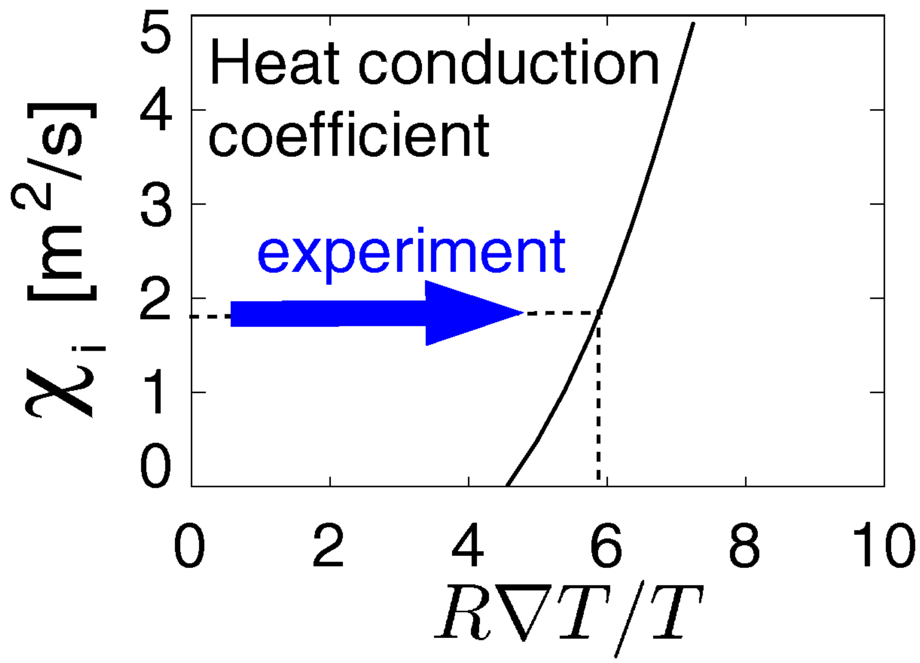
$$\frac{1}{L_T} = \left|\frac{\nabla T}{T}\right| > \frac{1}{L_{T,cr}}$$



ITG causes strong enhancement of radial transport

$$\frac{d\,ln\,T}{dr} = \frac{\nabla T}{T} = -\frac{1}{L_{T,cr}}$$

$$\int \frac{dT}{T} = \int_a^b -\frac{dr}{L_{T,cr}}$$



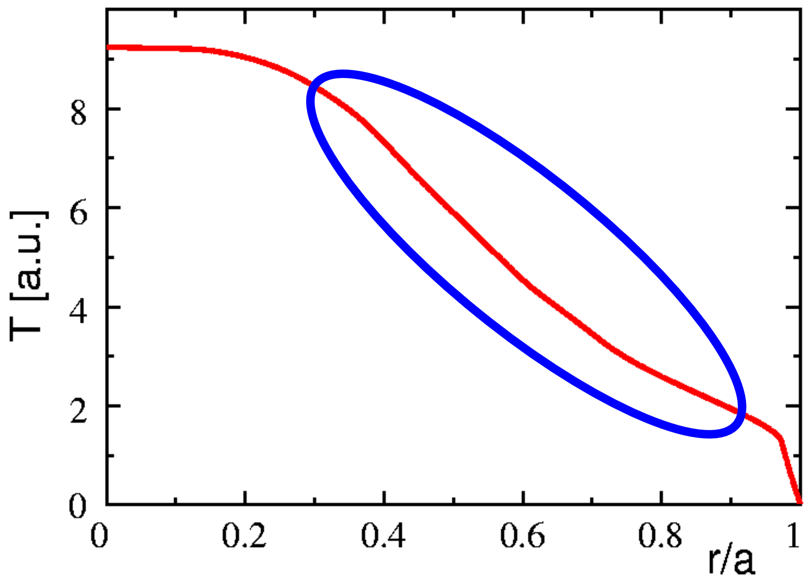
Turbulent transport rises with T gradient

$$\int \frac{dT}{T} = \int_a^b -\frac{dr}{L_{T,cr}}$$

$$lnT = -\frac{r}{L_{T,cr}}$$

$$lnT(b) - lnT(a) = \frac{a-b}{L_{T,cr}}$$

$$\ln\left(\frac{T(b)}{T(a)}\right) = \frac{a-b}{L_{T,cr}}$$



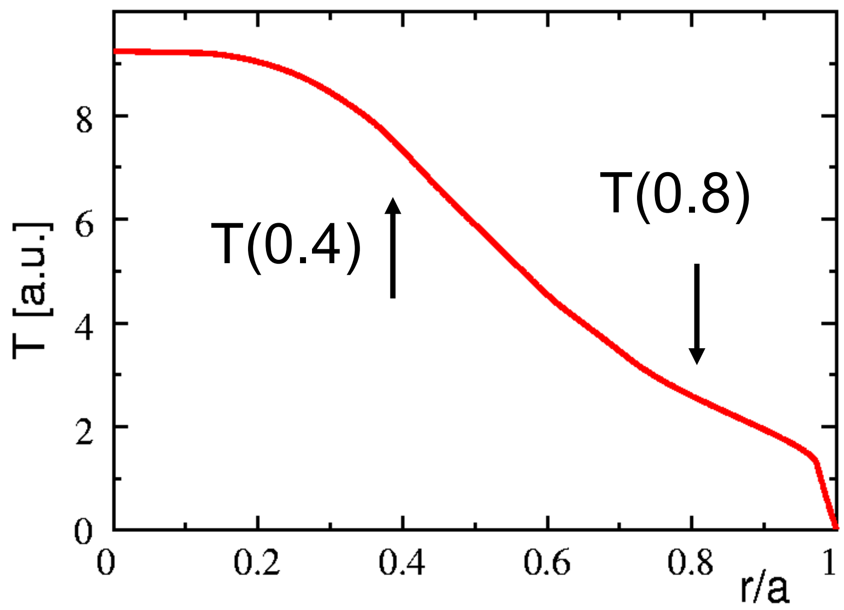
$$T(b) = T(a)e^{(a-b)/L_{T,cr}}$$

A certain critical T gradient cannot be exceeded – independent from heating power

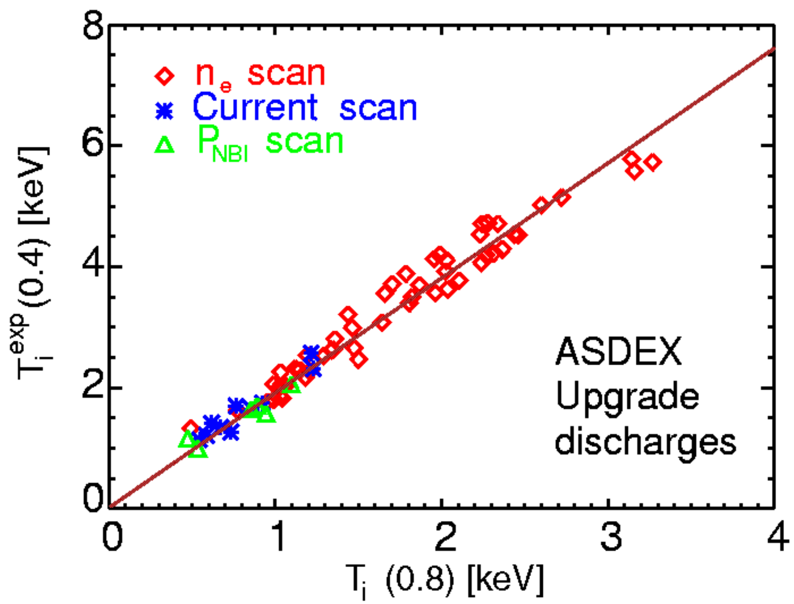
“stiff” temperature profiles

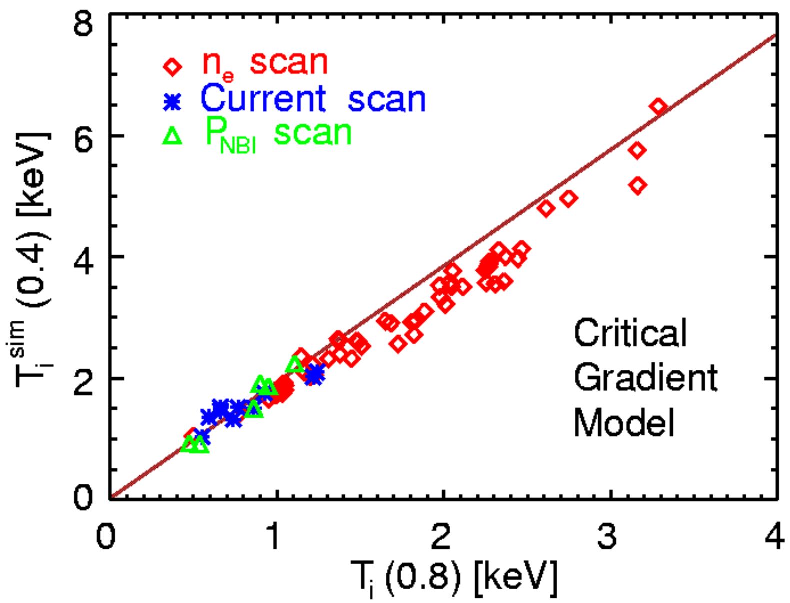
“stiff“ temperature profiles in theory and experiment

confirmed by experiment:
temperature at half radius is
proportional to edge temperature



modeling agrees with experiment

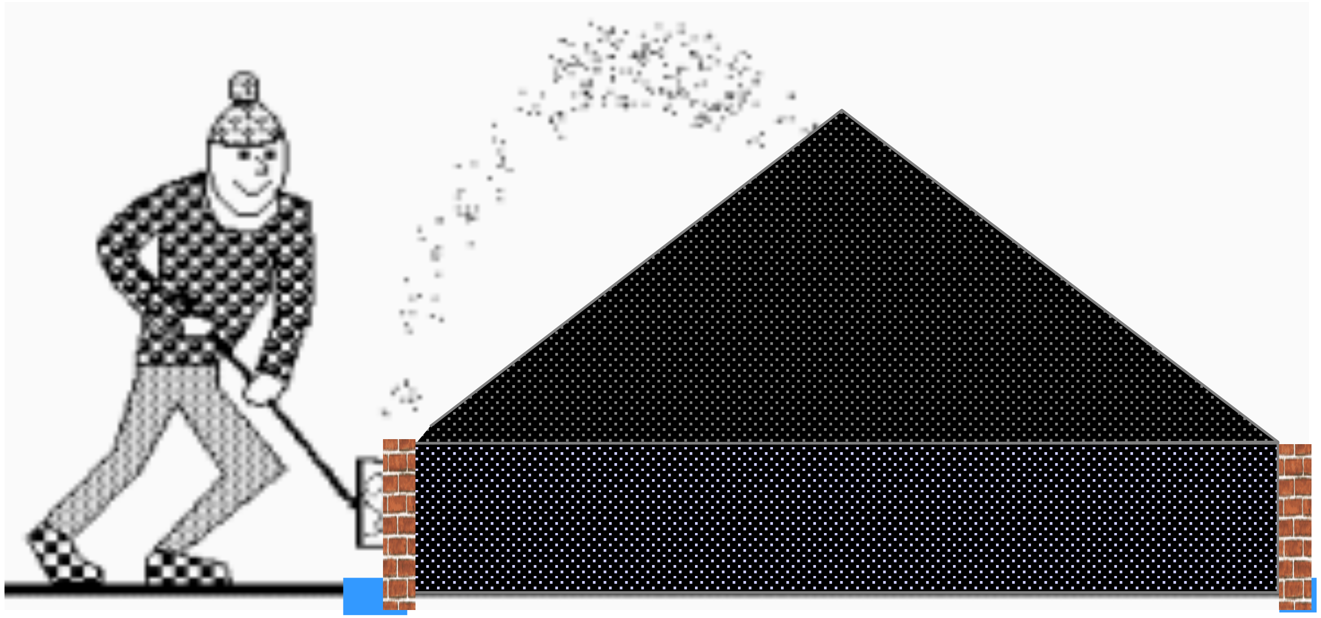




“Stiff” profiles and transport barriers

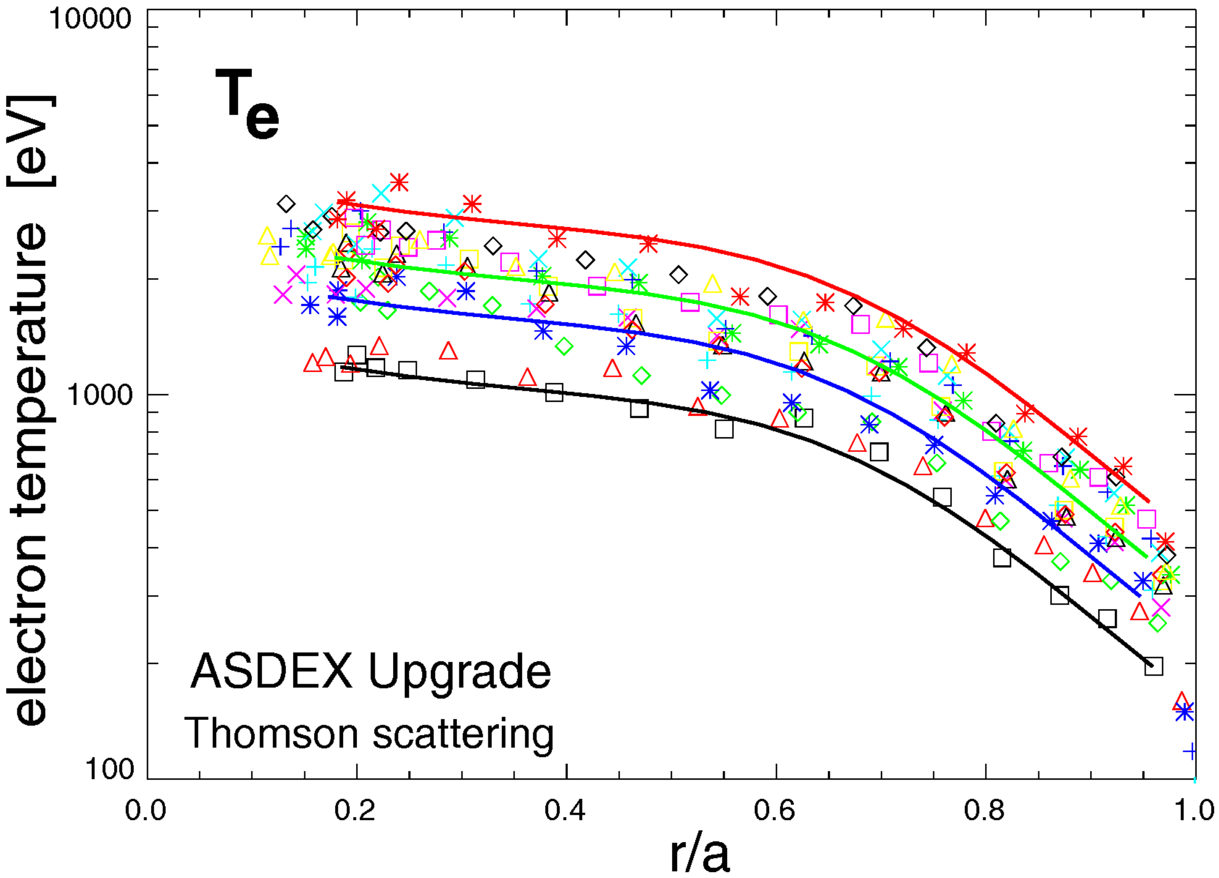
Turbulent transport limits (logarithmic) gradient of temperature profiles

Analogous to sand-pile: gradient limited



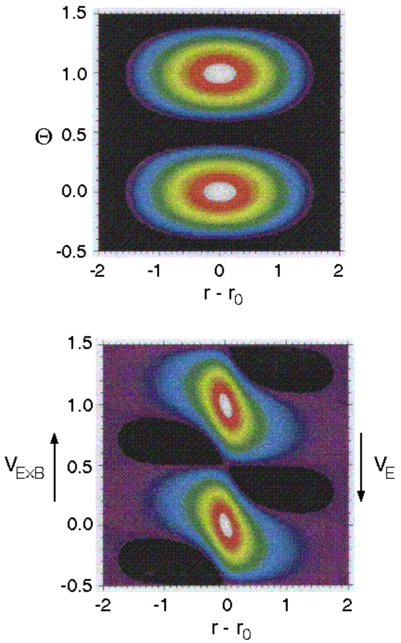
but height can be varied by “barriers”

Central temperature is determined by edge temperature



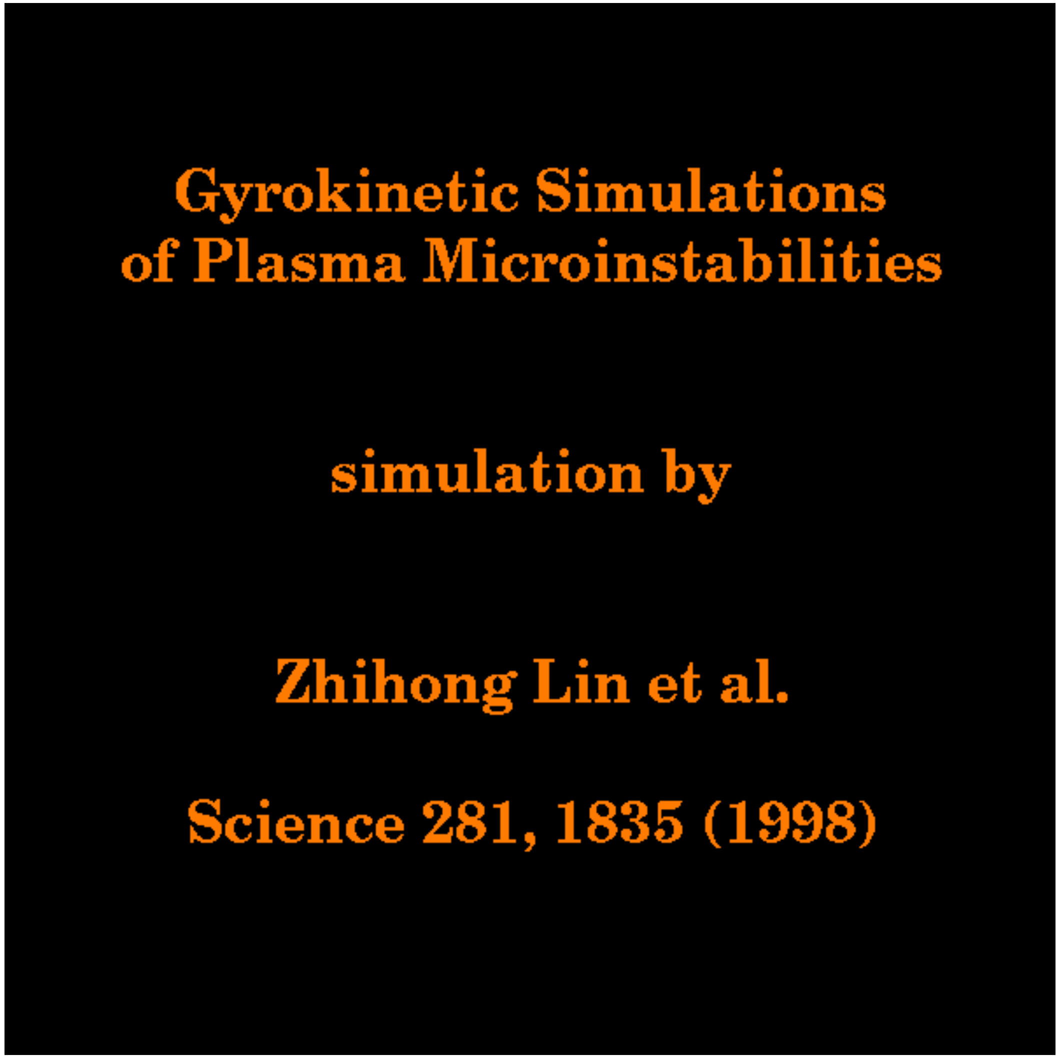
Turbulence is suppressed by sheared rotation

Macroscopic sheared rotation tilts eddies and tears them apart



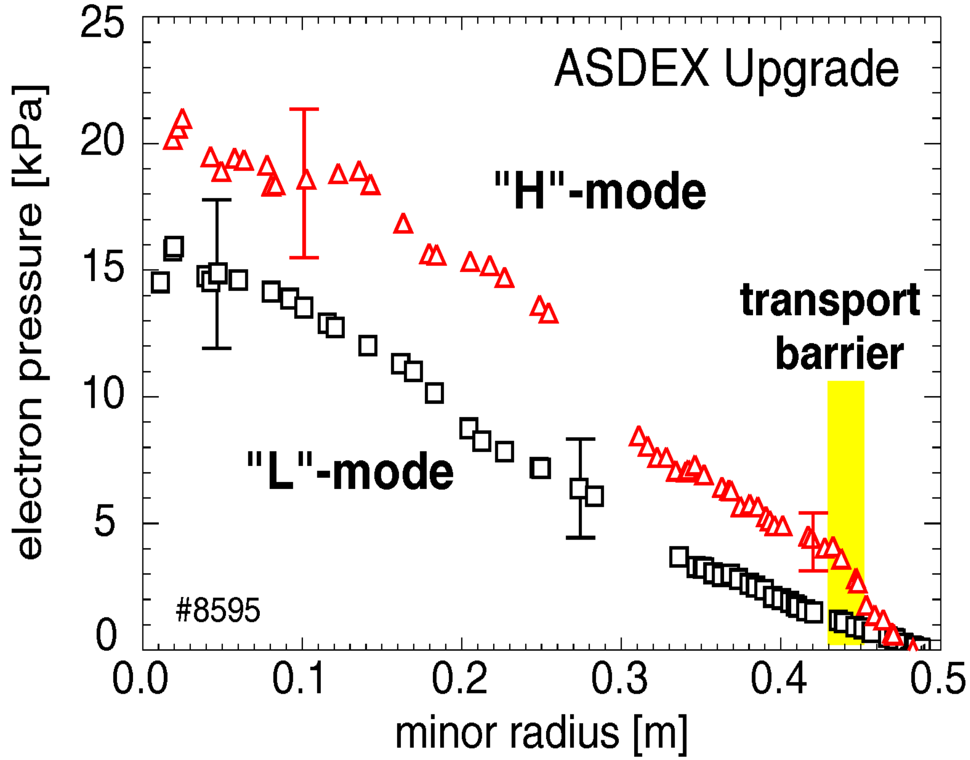
radial transport is proportional to eddy size

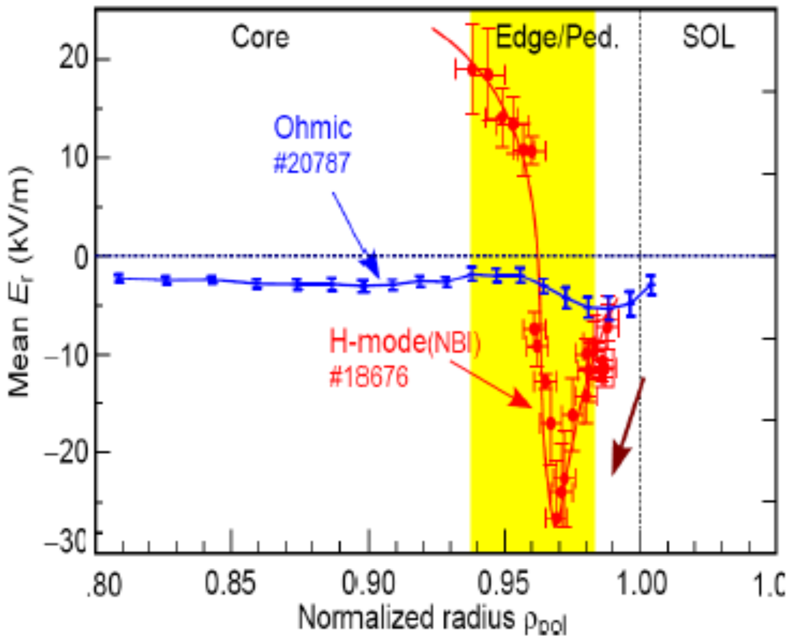
Sheared rotation is self generated (Reynolds stress)



Transport barriers due to turbulence suppression

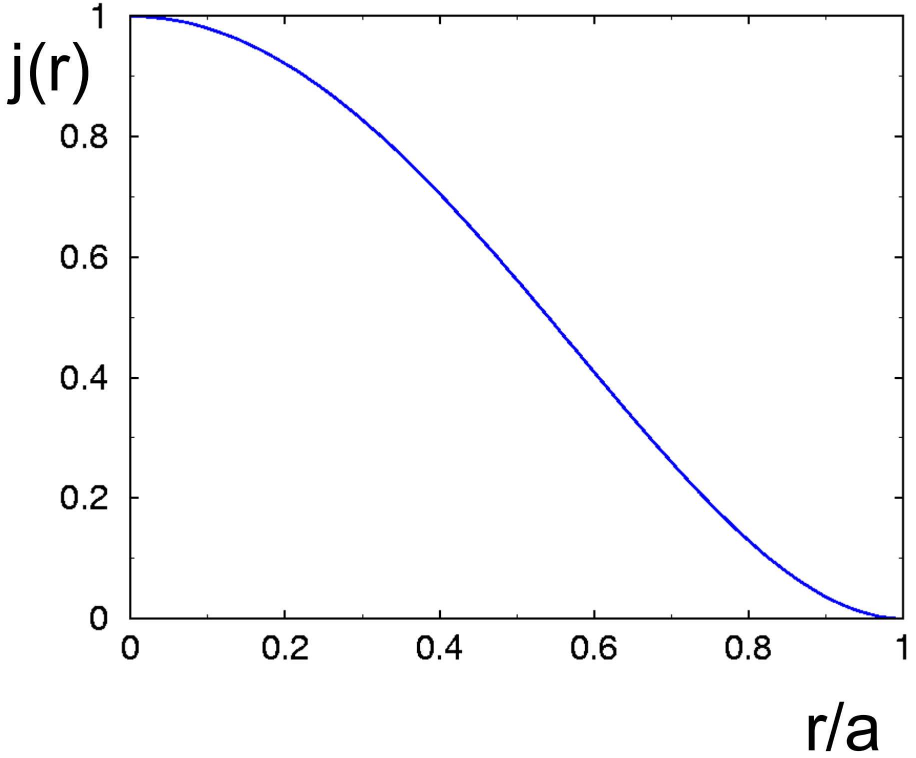
conventional Tokamak

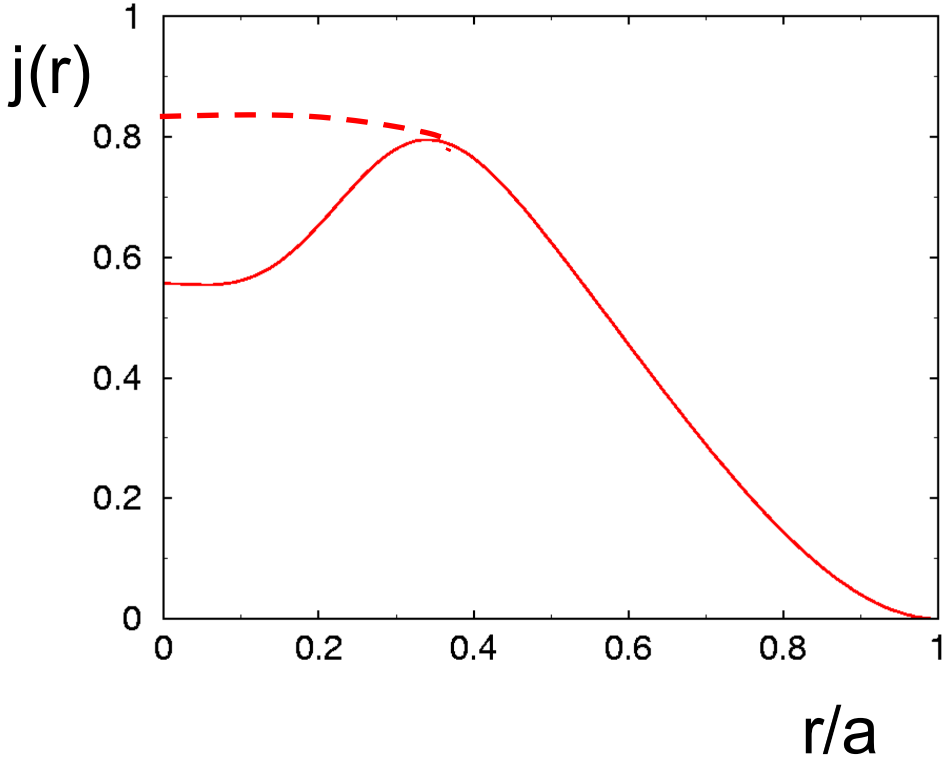




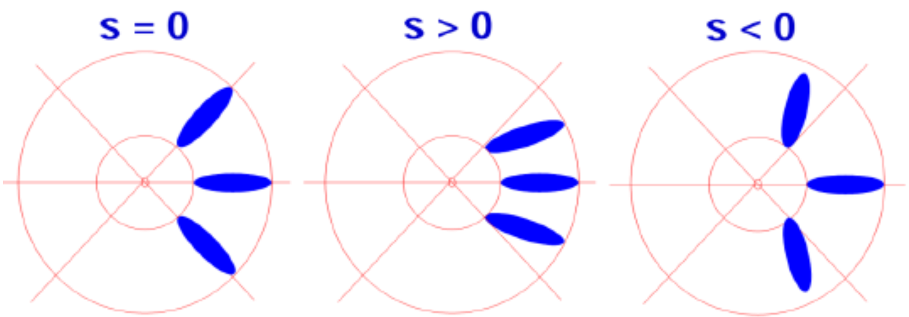
Turbulance suppression is most effective for non-monotonous current profiles

Standard j-profile

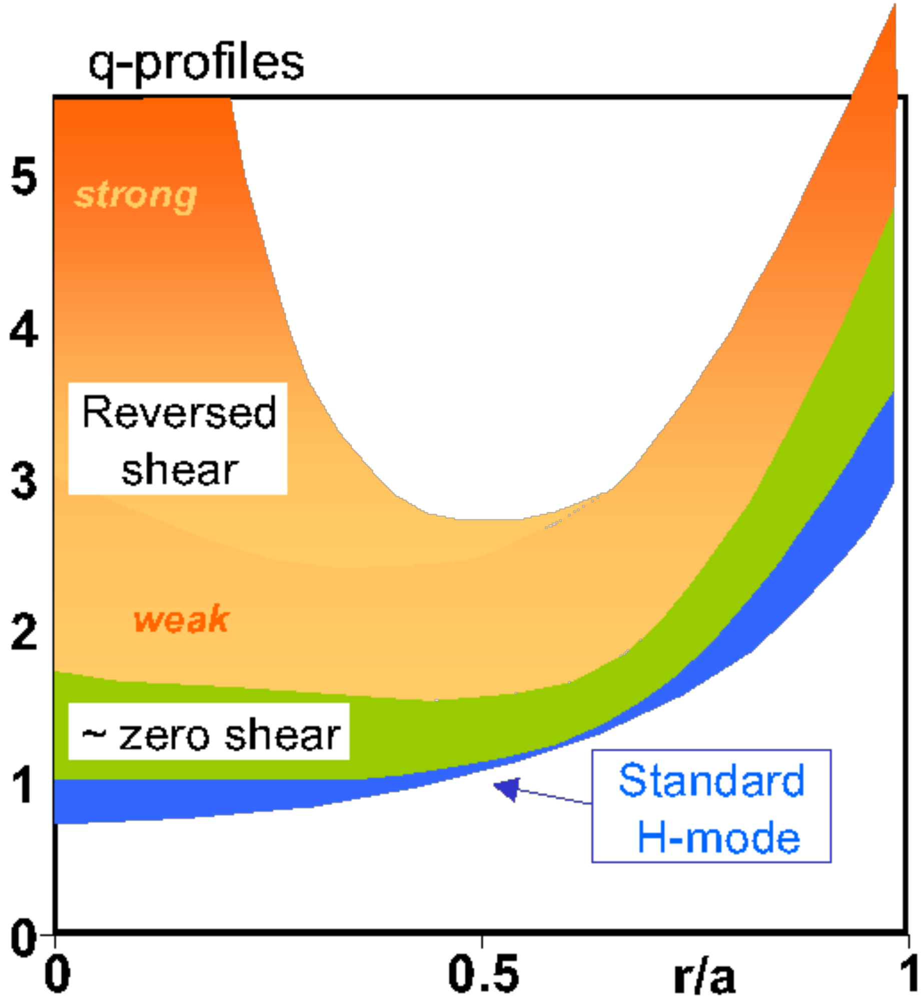




Turbulance suppression is most effective for non-monotonous current profiles

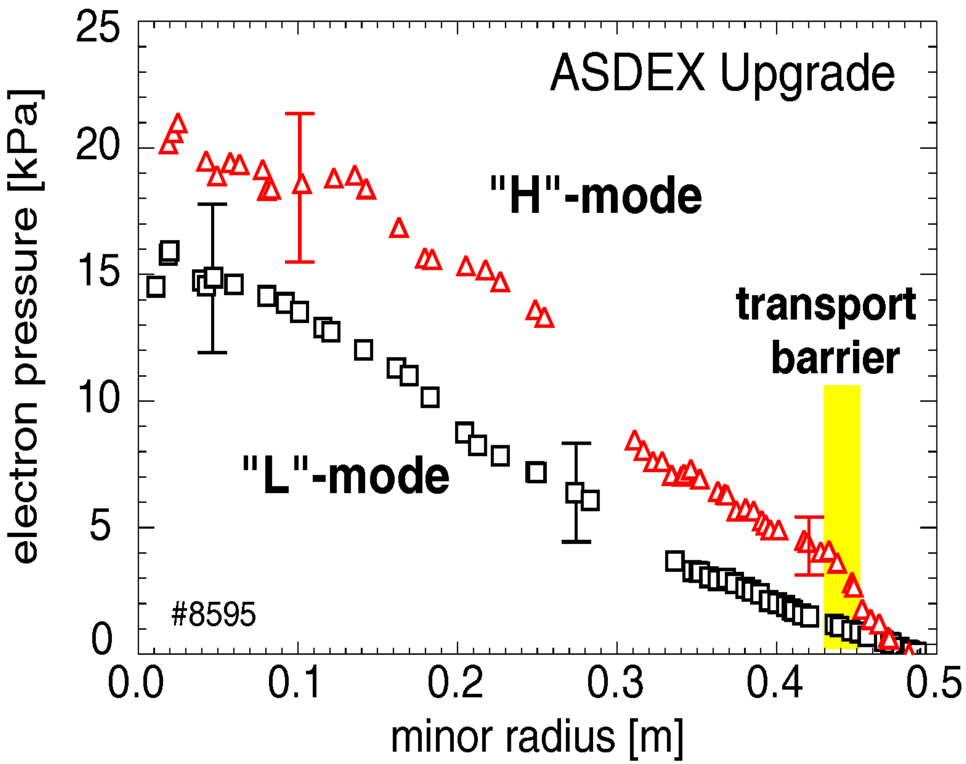


- Perturbations are field aligned, magnetic shear tilts the eddies and reduces the drive
- Maximum transport around s = 0.5

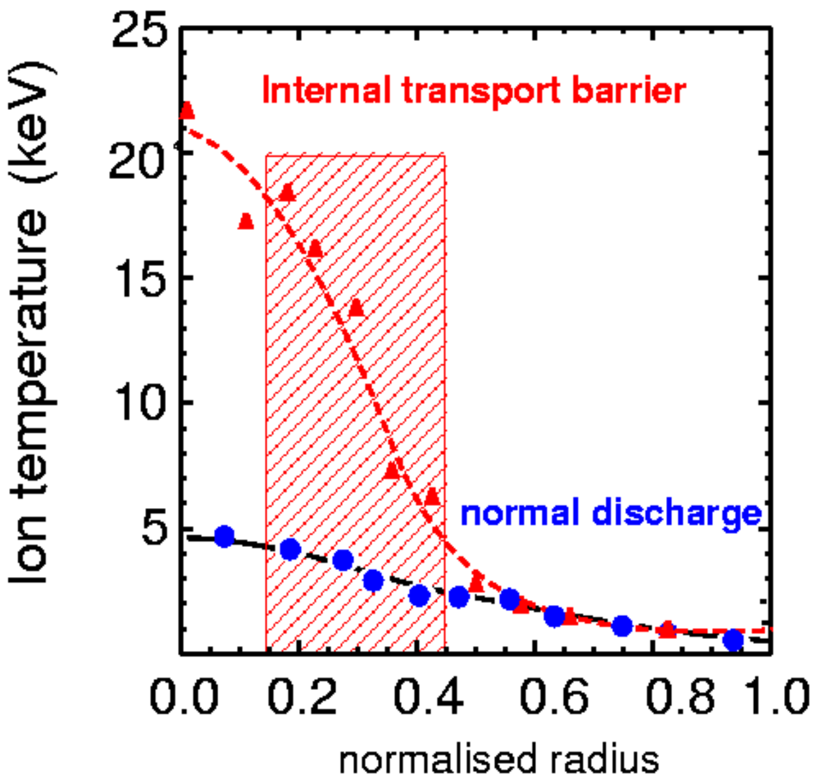


Transport barriers due to turbulence suppression

conventional Tokamak



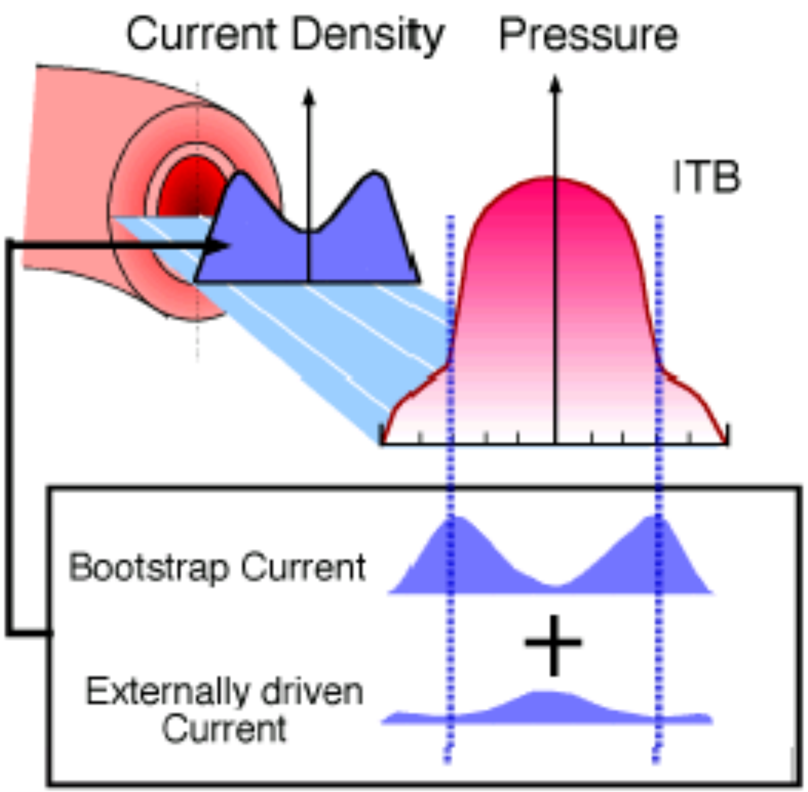
„Advanced Tokamak“



Ignition temperature at ASDEX Upgrade!

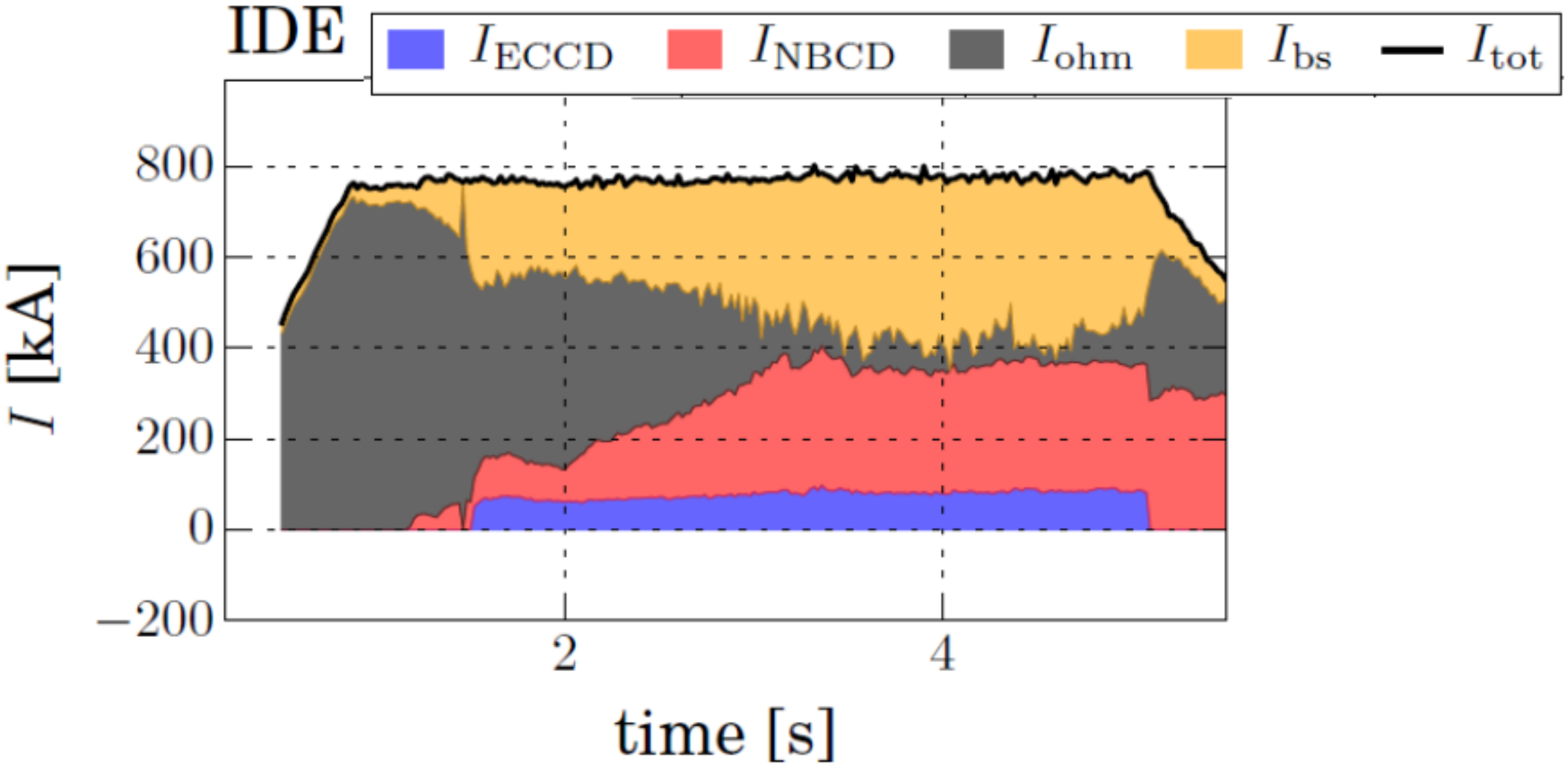
Advanced Tokamaks -perspectives

- Transport barierres →improved heat insulation
- ignition for smaller machines possible
- stationary operation due to non-inductive current drive



$$j_{BS} \sim \nabla p$$

Stationary Tokamaks – first results



State of the art:substitute simple scaling laws by prediction of density and temperature profiles